Bosonized Lagrangians in higher orders of the chiral expansion

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The derivation of effective meson Lagrangians in higher orders of the chiral expansion from bosonization of the four-quark interaction of the extended Nambu–Jona-Lasinio (NJL) model, treated as a local approximation of low-energy QCD, is reviewed. The calculated heat-kernel coefficients for the quark determinant of the bosonized NJL model through seventh order are presented in a systematic way. The results are used to fix the structure coefficients of the effective chiral Lagrangians in orders p^4 and p^6 of the momentum expansion. Various aspects of the use of bosonized Lagrangians to describe low-energy meson processes are discussed: the reduction of vector, axial-vector, and scalar resonances, regularization of graphs with meson loops, and the dependence of the phenomenological structure coefficients on the renormalization scheme and regularization parameters. Questions related to the physical justification of the NJL model as a low-energy limit of QCD are also discussed. Nonlocal corrections to the structure coefficients of bosonized Lagrangians are studied in this context. [S1063-7796(98)00201-0]

INTRODUCTION

The increased popularity in recent years of chiral Lagrangians and the calculation of meson amplitudes in higher orders of the momentum expansion is a consequence of the significant progress which has been made in the development of effective approaches to describing hadronic processes at low and intermediate energies, taking into account the dynamics of quarks and gluons at large distances. Obtaining a realistic theory of mesons and baryons as the low-energy limit of QCD is one of the fundamental problems of elementary-particle physics. Effective approaches are necessary because perturbative QCD becomes inapplicable at large distances, and broken chiral symmetry remains the basic dynamical principle governing strong interactions at low energies.

So far, no one has succeeded in deriving a chiral dynamics of hadrons from QCD in a rigorous mathematical sense. Nevertheless, the application of functional methods to approximate forms of QCD (see Refs. 1–13 and references therein) and QCD-motivated quark models (Refs. 14–24) arising as various extensions of the well known Nambu–Jona-Lasinio (NLJ) model[25] has led to considerable progress in our understanding of the relation between the broken chiral symmetry of strong hadronic interactions and the dynamics of quarks and gluons at large distances. In spite of its well known defects (nonrenormalizability and the absence of confinement), the NLJ model leads to a description in terms of nonperturbative QCD (the current and constituent quark masses, the quark condensate, the number of quark colors) of those properties of mesons and baryons which are determined by chiral symmetry and its breaking.[26–31] The bosonization of the NJL model also allows the derivation of meson Lagrangians with higher derivatives from the effective four-quark interaction[32–38] and thereby the fixing of the theoretically associated structure constants, which are treated as phenomenological parameters in the standard chiral perturbation theory.[39]

Despite the fact that the NJL model leads to a correct description of the low-energy phenomenology of mesons and baryons, it should be remembered that it is incomplete, primarily owing to the absence of quark confinement in it. The problem of nonrenormalizability is solved in the modern NJL model by introducing an appropriate ultraviolet cutoff. This procedure seems physically reasonable, because *a priori* it can be expected that the exclusion of large momentum transfers should not strongly affect the low-energy properties of quark bound states, owing to the rapid falloff of the gluon propagator in the region of asymptotic freedom. As a local low-energy approximation, the NJL model corresponds physically to the assumption that the properties of low-lying quark bound states (hadrons) are mainly determined by the range of momentum transfers lying between the regions of asymptotic freedom and quark confinement, where the momentum transfer is much smaller than the effective mass of the nonperturbative gluon. However, such dominance of the intermediate region is not completely obvious, owing to the pole behavior of the nonperturbative gluon propagator near zero momentum transfer. Therefore, at first glance the exclusion of the confinement region in the NJL model is not as natural a physical approximation as the introduction of an ultraviolet cutoff. This question requires special study, which has been done, in particular, in Ref. 23.

In this review we shall give a detailed discussion of the derivation of effective meson Lagrangians from bosonization of the extended NJL model, which can be treated as a local approximation in the low-energy limit of QCD. The first section of this review is devoted to the motivation of the NJL model. The extended NJL model not only includes symmetries corresponding to those of QCD and the explicit violation of the chiral symmetry by the current quark masses, but also reproduces the spontaneous breakdown of chiral symmetry associated with the appearance of a nonzero quark condensate and transformation of current quarks into constituent ones. In the bosonization approach, pseudoscalar, scalar, vector, and axial-vector mesons are introduced as collective fields corresponding to quark–antiquark bound states, and the interaction between these fields is described by the

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quark determinant arising in the integration over quarks in the generating functional.

In Secs. 2, 3, and 4 we discuss the technical details of calculating the quark determinant of the bosonized NJL model using the so-called heat-kernel expansion in powers of the "proper time" in the corresponding regularization. We present the calculated heat-kernel expansion coefficients through seventh order. A comparison is made with the analogous calculations of other groups. These results are used in Sec. 5 to fix the coupling constants of effective chiral Lagrangians, including terms of order p^6 of the momentum expansion. In this section we pay special attention to equivalent transformations and the equations of motion, which are used to eliminate terms with double derivatives and to reduce the effective Lagrangians to minimal form.

In this review we focus on the study of the effect of meson resonances on the description of the pseudoscalar sector. In Sec. 6 we study the effective pseudoscalar Lagrangian obtained by integration (reduction) of the vector, axial-vector, and scalar degrees of freedom in the generating functional of the bosonized NJL model. The reduction of heavy resonances, equivalent to including resonance exchange, is carried out using the static equations of motion arising from the generating functional after special chiral transformation of the meson fields. We discuss the higher-order corrections to the static equations of motion and the modification of the structure coefficients of the effective chiral Lagrangians after resonance reduction. Section 7 is devoted to the phenomenological aspects. There we study the description of low-energy meson processes in orders p^4 and p^6 of the chiral theory with bosonized Lagrangians.

As shown in the first section of this review, the NJL model arises as a local low-energy limit of QCD as a result of some sequence of rather crude approximations and auxiliary assumptions. Therefore, the predictive power of the NJL model, which is demonstrated in Sec. 7 for examples of pseudoscalar-meson interactions, in some sense even exceeds expectations. This is because the dynamics of meson processes at low energies is determined primarily by the broken chiral symmetry, which was incorporated from the start in the effective four-quark Lagrangian of the NJL model. In this sense the absence of confinement in this model should not affect those properties of mesons which are a consequence of chiral symmetry or its breakdown. In Sec. 8 we discuss the nonlocal corrections to the predictions of the NJL model treated in a bilocal approach as contributions arising from the pole behavior of the effective propagator of the nonperturbative gluon in the region of quark confinement. The approach developed in Refs. 11 and 23 allows us to obtain semiphenomenological estimates of the nonlocal corrections to the structure coefficients of bosonized chiral Lagrangians. It is shown that these corrections are small in order p^4 of the momentum expansion of the quark determinant of the bosonized NJL model, which is yet another argument in favor of this model as a realistic approximation for quark interactions at low energies.

In concluding the review we discuss possible directions for further development of effective quark models like the NJL model. We also briefly touch upon some of the questions and problems which we could not discuss in detail in this review.

1. QCD MOTIVATION OF THE NJL MODEL

Quantum chromodynamics is the generally accepted theory of the strong interaction in terms of quarks and gluons. It has been well confirmed experimentally in the perturbative regime at high energies ($E \gg 1$ GeV). Since quarks and gluons cannot be observed as free particles, it is assumed that they are bound to form colorless hadrons. There are many different phenomenological approaches and effective models for describing hadrons and their interactions at low and intermediate energies ($E < 1$ GeV). However, so far it has not been possible to derive a meson theory from QCD in a mathematically rigorous fashion taking into account quark and gluon confinement. The difficulties arising here are largely due to the unknown behavior of the QCD Green functions at low energies. Nevertheless, significant progress has been achieved in this area, owing to effective bilocal approaches which in the local limit lead to the NJL model, the bosonization of which allows effective chiral Lagrangians describing meson strong interactions to be obtained. In this section we constructively study the bilocal approach and the physical approximations taking us from QCD to effective quark models within this formalism.

Strong interactions are described in the Minkowski metric by the QCD action with the color group $SU(N_c)$:

$$\mathcal{S}[\bar{q},q,A]=\int d^4x\left[\bar{q}(i\hat{D}-m_0)q-\frac{1}{4}G^a_{\mu\nu}G^{a\mu\nu}\right], \quad (1.1)$$

corresponding to the interaction of $n\times N_c$ colored current quarks q with (N_c^2-1) gluon fields A^a_μ (n is the number of flavors). In Eq. (1.1) there is understood to be an implicit summation over color indices, and also a summation over repeated Dirac and flavor indices. The generators of the $U(n)$ flavor group λ^a are normalized as

$$\mathrm{tr}\,\lambda^a\lambda^b=2\delta^{ab},\quad a,b=0,\ldots,n^2-1;\quad \lambda_0=\sqrt{2/n}\,\mathbf{1};$$

$m_0=\mathrm{diag}(m_0^1,\ldots,m_0^n)$ is the mass matrix of the bare (current) quarks and explicitly violates the chiral and diagonal $U(n)$ symmetry. The covariant derivative is defined as

$$D_\mu=\partial_\mu-ig\,\frac{\lambda^a_c}{2}A^a_\mu,$$

where λ^a_c are the generators of the color group, and the gluon field-strength tensor has the form

$$G^a_{\mu\nu}=\partial_\mu A^a_\nu-\partial_\nu A^a_\mu-gf_{abc}A^b_\mu A^c_\nu,$$

where g is the QCD coupling constant and f_{abc} are the structure constants of the $SU(3)$ group. We have also used the notation $\hat{D}$ for the contraction $\gamma_\mu D^\mu$.

The generating functional of QCD in the absence of gluon sources has the form

$$\mathcal{Z}[\xi,\bar{\xi}]=\int \mathcal{D}\bar{q}\mathcal{D}q\mathcal{D}A\,\exp\Bigg(i\mathcal{S}[\bar{q},q,A]+i\int d^4x(\bar{q}\xi+\bar{\xi}q)\Bigg), \quad (1.2)$$

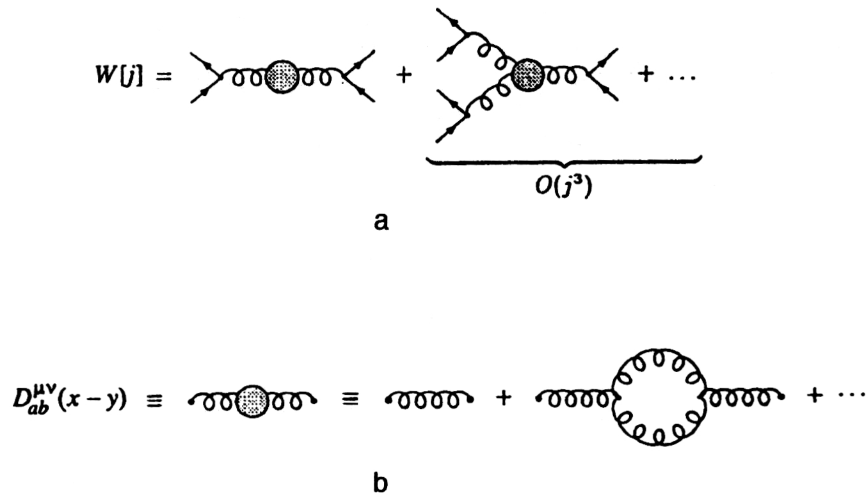

FIG. 1. Graphical representation of the expansion of (a) the gluon generating functional $W[j]$, and (b) the complete gluon propagator $D^{\mu\nu}_{ab}(x-y)$.

where ξ and $\bar{\xi}$ are external sources of the quark fields. The integral (1.2) is antiperiodic in the quark and ghost fields, but periodic in the gluon fields. The Faddeev–Popov ghost fields and gauge-fixing terms are included in the gluon measure.

We introduce into the functional integral over gluon fields a term involving external sources, the role of which is played by flavor-singlet local currents

$$j^a_\mu(x)=\bar{q}(x)\gamma_\mu\frac{\lambda^a_c}{2}q(x)$$

associated with the gluon fields A^a_μ. Then the QCD generating functional is rewritten identically as

$$\mathscr{Z}[\xi,\bar{\xi}]=\left|\int\mathscr{D}\bar{q}\mathscr{D}q\exp\left[i\int d^4x\bar{q}\left(i\hat{\partial}-m_0+g\frac{\lambda^a_c}{2}\gamma_\mu\frac{\delta}{\delta j^a_\mu}\right)q+i\int d^4x(\bar{q}\xi+\bar{\xi}q)\right]\int\mathscr{D}A\exp\left(-\frac{i}{4}\int d^4xG^a_{\mu\nu}G^{a\mu\nu}+i\int d^4xj^a_\mu A^{a\mu}\right)\right|_{j=0},$$

and after integration over the gluon fields we obtain[1,2,10]

$$\mathscr{Z}=\int\mathscr{D}\bar{q}\mathscr{D}q\exp\left[i\int d^4x\bar{q}(x)(i\hat{\partial}-m_0)q(x)\right]\exp(iW[j]),\tag{1.3}$$

where we have discarded the normalization factors and external quark sources. The gluon generating functional $W[j]$ in Eq. (1.3) can be written as an expansion (Fig. 1a)

$$W[j]=\frac{1}{2}\int d^4x\,d^4yj^a_\mu(x)D^{ab}_{\mu\nu}(x-y)j^b_\nu(y)=O(j^3).\tag{1.4}$$

Here

$$D^{a,b}_{\mu\nu}(x-y)=\int_{\text{connected}}\mathscr{D}AA^a_\mu(x)A^b_\nu(y)\times\exp\left(-\frac{i}{4}\int d^4xG^a_{\mu\nu}G^{a\mu\nu}\right)$$

is the exact gluon propagator including all the gluon self-interactions and interactions of gluons with ghost fields (Fig. 1b), with quark loops excluded, and

$$O(j^3)=\sum_{n=3}^{\infty}\frac{1}{n!}\int d^4x_1\ldots d^4x_n\times D^{a_1\ldots a_n}_{\mu_1\ldots\mu_n}(x_1,\ldots,x_n)\prod_{i=1}^{n}j^{a_i}_{\mu_i}(x_i)$$

is the contribution of gluon vertices of order higher than three. The n-point Green functions $D^{a_1\ldots a_n}_{\mu_1\ldots\mu_n}$ introduced in this manner contain complete information about the gluon dynamics.

Restricting ourselves to one-gluon exchange (the first graph in Fig. 1a) in the expansion of $W[j]$ (1.4) and discarding higher-order gluon vertices, we obtain an expression for the generating functional corresponding to an effective low-energy approximation to QCD of the form

$$\mathscr{Z}=\int\mathscr{D}\bar{q}\mathscr{D}q\exp\left\{i\left[\int d^4x\bar{q}(x)(i\hat{\partial}-m_0)q(x)\right]+i\mathscr{S}_{\text{int}}\right\}.$$

Here

$$\mathscr{S}_{\text{int}}=-i\frac{g^2}{2}\int\int d^4x\,d^4yj^{a\mu}(x)D^{ab}_{\mu\nu}(x-y)j^{b\nu}(y)\tag{1.5}$$

is the effective action corresponding to quark interaction via nonperturbative gluon exchange, and

$$g^2D^{ab}_{\mu\nu}(x)=\delta^{ab}g_{\mu\nu}D(x)=\delta^{ab}g_{\mu\nu}4\pi\int\frac{d^4q}{(2\pi)^4}\frac{\alpha(q^2)}{q^2}e^{iqx}$$

is the nonperturbative gluon propagator in the Feynman gauge. In this representation of the propagator it is assumed that all the properties of the quark interaction via gluon exchange are determined by the properties of the running coupling constant $\alpha(q^2)$. The exact form of the nonperturbative gluon propagator is determined by the unknown gluon dynamics at large distances.

After a Fierz transformation, the effective action (1.5) becomes

$$S_{\rm int}=\frac{i}{2}\int\int d^4x\, d^4y D(x-y)\bar{q}(x)\frac{\mathcal{M}^\theta}{2}q(y)\bar{q}(y)\frac{\mathcal{M}^\theta}{2}q(x), \quad (1.6)$$

where $\mathcal{M}^\theta$ is the tensor product of Dirac, flavor, and color matrices of the form

$$\left\{\mathbf{1},i\gamma_5,i\sqrt{\frac{1}{2}}\gamma^\mu,i\sqrt{\frac{1}{2}}\gamma_5\gamma^\mu\right\}^D\left\{\frac{1}{2}\lambda^a\right\}^F\left\{\frac{4}{3}\mathbf{1}\right\}^C. \quad (1.7)$$

Here we shall study the $SU(3)$ flavor group and restrict ouselves to the contributions of only color singlets $\bar{q}q$.

Since the exact behavior of the Green functions $D(x)$ at large distances is unknown, any model ansatz can be used for it. For example, owing to condensation at low energies,[45] the nonperturbative gluon acquires a nonzero mass related to the gluon condensate as[21]

$$m_G^2=\frac{15}{32}\left[16\pi^2\left\langle\frac{g^2}{4\pi^2}(G^a_{\mu\nu})^2\right\rangle\right]^{1/2}.$$

Using the value of the gluon condensate

$$\left\langle\frac{g^2}{4\pi^2}(G^a_{\mu\nu})^2\right\rangle=(410\pm 80\ \text{MeV})^2,$$

extracted from processes $e^+e^-\to$hadrons (Ref. 46), we obtain the estimate $m_G^2=(806\pm 275\ \text{MeV})^2$ for the nonperturbative gluon mass. In the momentum-transfer range $0\leqslant q^2\leqslant m_G^2$ intermediate between the confinement region and the region of quark asymptotic freedom, the gluon propagator in momentum space can be approximated as a constant:

$$D(q)=\frac{1}{q^2-m_G^2}\approx-\frac{1}{m_G^2},$$

which in coordinate space will correspond to a local ansatz $D(x)\sim\delta^{(4)}(x)$. Then the effective action (1.6) will lead to an effective four-quark interaction described by the Lagrangian of the NJL model:[25]

$$\mathcal{L}_{\rm int}=2G_1\left[\left(\bar{q}\frac{\lambda^a}{2}q\right)^2+\left(\bar{q}i\gamma_5\frac{\lambda^a}{2}g\right)^2\right]-2G_2\left[\left(\bar{q}\gamma_\mu\frac{\lambda^a}{2}q\right)^2+\left(\bar{q}\gamma_5\gamma_\mu\frac{\lambda^a}{2}q\right)^2\right], \quad (1.8)$$

where the universal coupling constants G_1 and G_2 in this case will be related as $G_1=2G_2$.

This approximation neglects the property of quark confinement associated with the singular pole behavior of the gluon propagator in momentum space near zero momentum transfer (the confinement region). It also neglects the rapid falloff of the gluon propagator $D(q)$ in the region of quark asymptotic freedom. Nevertheless, the bosonization of the NJL model leads to effective chiral Lagrangians which ensure a good description of low-energy meson processes both in the leading and in higher orders of the chiral theory.

2. BOSONIZATION OF THE NJL MODEL

The NJL model, whose bosonization we shall discuss in this section, not only incorporates all the needed flavor symmetries determining the quark dynamics in low-energy QCD, but also provides a simple scheme for spontaneous breakdown of chiral symmetry: explicit breaking by massive quark terms. In this scheme current quarks become constituent quarks, owing to the appearance of a nonzero quark condensate. Light constituent pseudoscalar Nambu–Goldstone bosons arise together with heavier dynamical vector and axial-vector mesons with the correct relative weights arising from renormalization.

The approach under discussion is based on the effective four-quark Lagrangian of the strong interactions of the extended NJL model, invariant under the global color symmetry $SU(N_c)$ and the $SU(3)_L\otimes SU(3)_R$ flavor symmetry:

$$\mathcal{L}_{\rm NJL}=\bar{q}(i\hat{\partial}-m_0)q+\mathcal{L}_{\rm int}, \quad (2.1)$$

where the interacting part of the Lagrangian $\mathcal{L}_{\rm int}$ is given by (1.8). It is interesting to note that the group structures of QCD,

$$SU(3)_c^{\rm local}\otimes SU(N_f)_L\otimes SU(N_f)_R\otimes U(1)\otimes S,$$

and of the NJL model,

$$SU(3)_c^{\rm global}\otimes SU(N_f)_L\otimes SU(N_f)_R\otimes U(1)\otimes S,$$

are very similar (here S represents a set of discrete symmetries such as C-, P-, and T-conjugation).

Using the standard approach to the bosonization of quark interactions based on functional integration, we can obtain the effective meson action from the Lagrangian of the NJL model (2.1). First, in the usual manner[1,2,10] we introduce collective meson fields corresponding to scalar (S), pseudoscalar (P), vector (V), and axial-vector (A) mesons associated with the quark bilinear combinations

$$S^a=-4G_1\bar{q}\frac{\lambda^a}{2}q,\quad P^a=-4G_1\bar{q}i\gamma^5\frac{\lambda^a}{2}q,$$

$$V^a_\mu=i4G_2\bar{q}\gamma_\mu\frac{\lambda^a}{2}q,\quad A^a_\mu=i4G_2\bar{q}\gamma_\mu\gamma^5\frac{\lambda^a}{2}q. \quad (2.2)$$

After substituting these expressions into (2.1), the Lagrangian of the NJL model can be rewritten in an equivalent form:

$$\mathcal{L}_{\rm NJL}=-\frac{1}{4G_1}\text{tr}(\Phi^\dagger\Phi)-\frac{1}{4G_2}\text{tr}(V_\mu^2+A_\mu^2)+\bar{q}i\hat{\mathbf{D}}q, \quad (2.3)$$

where the trace runs over the flavor indices and $\hat{\mathbf{D}}$ is the Dirac operator in the presence of collective meson fields:

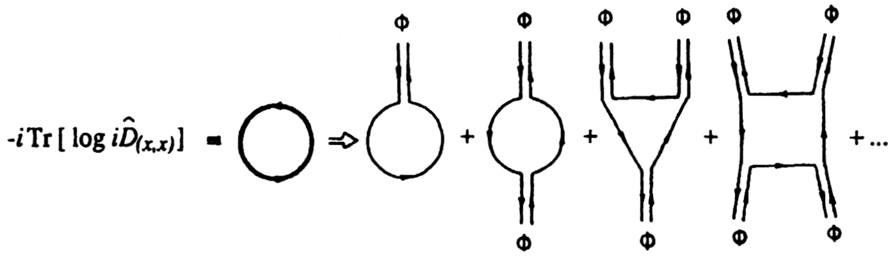

FIG. 2. Graphical representation of the expansion of the quark determinant in quark fields with external collective fields.

$$i\hat{\mathbf{D}}=i(\hat{\partial}+\hat{V}+\hat{A}\gamma^5)-P_R(\Phi+m_0)-P_L(\Phi^\dagger+m_0)$$
$$=[i(\hat{\partial}+\hat{A}^R)-(\Phi+m_0)]P_R+[i(\hat{\partial}+\hat{A}^L)-(\Phi^\dagger+m_0)]P_L. \quad (2.4)$$

Here $\Phi=S+iP$, $\hat{V}=V_\mu\gamma^\mu$, and $\hat{A}=A_\mu\gamma^\mu$; $P_{R/L}=\frac{1}{2}(1\pm\gamma_5)$ are the chiral right/left projection operators; $\hat{A}^{R/L}=\hat{V}\pm\hat{A}$ are right/left combinations of fields, and

$$S=S^a\frac{\lambda^a}{2},\quad P=P^a\frac{\lambda^a}{2},\quad V_\mu=-iV_\mu^a\frac{\lambda^a}{2},$$
$$A_\mu=-iA_\mu^a\frac{\lambda^a}{2}.$$

Let us consider the Green-function generating functional corresponding to the Lagrangian (2.3). Since it is bilinear in the quark fields, one can integrate over them, after which the generating functional takes the form (for simplicity we drop the quark sources)

$$\mathscr{Z}=\int\mathscr{D}\Phi\mathscr{D}\Phi^\dagger\mathscr{D}V\mathscr{D}A\,\exp[i\mathscr{S}(\Phi,\Phi^\dagger,V,A)], \quad (2.5)$$

where

$$\mathscr{S}(\Phi,\Phi^\dagger,V,A)=\int d^4x\left[-\frac{1}{4G_1}\,\mathrm{tr}(\Phi^\dagger\Phi)-\frac{1}{4G_2}\,\mathrm{tr}(V_\mu^2+A_\mu^2)\right]-i\,\mathrm{Tr}'[\ln(i\hat{\mathbf{D}})] \quad (2.6)$$

is the effective action for scalar, pseudoscalar, vector, and axial-vector mesons. The trace Tr′ is taken over the space-time, color, flavor, and Dirac indices:

$$\mathrm{Tr}'=\int d^4x\,\mathrm{Tr},\quad \mathrm{Tr}=\mathrm{tr}_\gamma\cdot\mathrm{tr}_C\cdot\mathrm{tr}.$$

The first term in (2.6), which is quadratic in the meson fields, arises as a result of linearization of the four-quark interaction. The second term is the quark determinant describing the meson interaction.

The quark determinant can be calculated either by making an expansion in quark loops with external collective meson fields[14–16] (see Fig. 2), or by the method of heat-kernel coefficients in proper-time regularization.[47,48] The modulus of the quark determinant contributes to the nonanomalous part of the effective Lagrangian, while its complex phase determines the anomalous effective Wess–Zumino action[59] associated with chiral anomalies.

The following representation of the field Φ corresponds to nonlinear parametrization of the chiral symmetry:

$$\Phi=\Omega\Sigma\Omega. \quad (2.7)$$

The matrix of the scalar fields $\Sigma(x)$ belongs to the diagonal flavor group, while the matrix $\Omega(x)$ represents the pseudoscalar degrees of freedom φ remaining in the $U(n)_L\times U(n)_R/U_V(n)$ space. The matrix $\Omega(x)$ can be parametrized by a unitary matrix

$$\Omega(x)=\exp\left(\frac{i}{\sqrt{2}F_0}\,\varphi(x)\right),\quad \varphi(x)=\varphi^a(x)\frac{\lambda^a}{2},$$

where F_0 is the bare value of the $\pi\to\mu\nu$ decay constant. Under chiral rotations

$$q\to\tilde{q}=(P_L\xi_L+P_R\xi_R)q$$

the fields Φ and $A_\mu^{R/L}$ transform as

$$\Phi\to\tilde{\Phi}=\xi_L\Phi\xi_R^\dagger,$$
$$A_\mu^R\to\tilde{A}_\mu^R=\xi_R(\partial_\mu+A_\mu^R)\xi_R^\dagger,\quad A_\mu^L\to\tilde{A}_\mu^L=\xi_L(\partial_\mu+A_\mu^L)\xi_L^\dagger, \quad (2.8)$$

The electromagnetic interaction of mesons with the photon field $\mathscr{A}_\mu$ is introduced via the substitution

$$V_\mu\to V_\mu+ieQ\mathscr{A}_\mu,$$

where Q is the matrix of quark electric charges.

The integral (2.5), which now depends on $\Omega(x)$ and $\Sigma(x)$, can be rewritten as

$$\mathscr{Z}=\int\mathscr{D}\mu(\Omega\Sigma)\mathscr{D}V\mathscr{D}A\,\exp\left\{i\int d^4x\left[-\frac{1}{4G_1}\,\mathrm{tr}\,\Sigma^2-\frac{1}{4G_2}\,\mathrm{tr}(V_\mu^2+A_\mu^2)\right]\right\}\cdot\det(i\hat{\mathbf{D}}(\Omega,\Sigma,V,A),$$

where $\mathscr{D}\mu(\Omega\Sigma)$ is the integration measure for transformations of the fields (2.7).

In the simplest case, when $\Omega=1$ and $A_\mu^{R/L}=0$, the mean-field equation for the Σ field reduces to a Schwinger–Dyson equation of the form

$$\Sigma_0=i2G_1N_c\,\mathrm{tr}_\gamma\frac{1}{i\hat{\mathbf{D}}^{\Omega=1}}. \quad (2.9)$$

In the approximation of vanishing current quark masses this equation always has the trivial solution $\Sigma_0=0$ corresponding to the chirally symmetric phase. However, for a value of G_1 exceeding some critical value, the solution $\Sigma_0=0$ becomes unstable and a new vacuum with nonzero Σ_0 appears. This corresponds to the phase of spontaneously broken chiral symmetry (the Nambu–Goldstone phase).

Assuming that the solution Σ_0 has diagonal form

$$\Sigma_0=\text{diag}(\sigma_1^0,\sigma_2^0,\ldots,\sigma_n^0),$$

from (2.9) we obtain the gap equation

$$\sigma_i^0=-i\,\frac{8G_1N_c}{(2\pi)^4}\int^\Lambda d^4k\,\frac{\sigma_i^0+m_i^0}{k^2-(\sigma_i^0+m_i^0)^2}$$
$$\equiv-2G_1\langle\bar{q}_iq_i\rangle, \tag{2.10}$$

where $\langle\bar{q}q\rangle$ is the quark condensate and Λ is the cutoff parameter. Using (2.10) and assuming approximate flavor symmetry of the quark condensate, we obtain

$$\sigma_i^0=-2G_1\langle\bar{q}_iq_i\rangle\equiv\mu.$$

In what follows we shall always (except in Sec. 6) neglect the quantum fluctuations of the scalar field Σ about its vacuum expectation value μ (i.e., we take $\Sigma\sim\Sigma_0\sim\mu\mathbf{1}$, where μ is the constituent quark mass averaged over flavor).

3. CALCULATION OF THE QUARK DETERMINANT BY THE HEAT-KERNEL METHOD

Let us begin our study of the heat-kernel method by calculating the nonanomalous part of the effective action corresponding to the modulus of the quark determinant, which in proper-time (τ) regularization is defined as the integral

$$\ln|\det i\hat{\mathbf{D}}|=-\frac{1}{2}\,\text{Tr}'\,\ln(\hat{\mathbf{D}}^\dagger\hat{\mathbf{D}})$$
$$=-\frac{1}{2}\int_{1/\Lambda^2}^{\infty}d\tau\,\frac{1}{\tau}\,\text{Tr}'\,K(\tau), \tag{3.1}$$

where $K(\tau)=e^{-\mathbf{A}\tau}$ is the so-called heat kernel for the operator $\mathbf{A}=\hat{\mathbf{D}}^\dagger\hat{\mathbf{D}}$ (Ref. 53), and Λ is an internal regularization parameter coinciding with the cutoff parameter in Eq. (2.10). The full heat kernel satisfies the heat-conduction equation

$$\frac{\partial}{\partial\tau}K(\tau)+\mathbf{A}K(\tau)=0$$

with the boundary condition $K(\tau=0)=1$.

In our case of the NJL model the operator $\mathbf{A}$ can be written as

$$\mathbf{A}=d_\mu d^\mu+a(x)+\mu^2,$$

where

$$d_\mu=\partial_\mu+\Gamma_\mu,\quad \Gamma_\mu=V_\mu+A_\mu\gamma^5,$$

$$a(x)=i\hat{\nabla}H+H^\dagger H+\frac{1}{4}[\gamma^\mu,\gamma^\nu]\Gamma_{\mu\nu}-\mu^2.$$

Here we use the following notation:

$$H=P_R\Phi+P_L\Phi^\dagger=S+i\gamma_5P,$$

$$\Gamma_{\mu\nu}=[d_\mu,d_\nu]=\partial_\mu\Gamma_\nu-\partial_\nu\Gamma_\mu+[\Gamma_\mu,\Gamma_\nu]$$
$$=F_{\mu\nu}^V+\gamma^5F_{\mu\nu}^A,$$

where $F_{\mu\nu}^{V,A}$ are the field-strength tensors;

$$F_{\mu\nu}^V=\partial_\mu V_\nu-\partial_\nu V_\mu+[V_\mu,V_\nu]+[A_\mu,A_\nu],$$

$$F_{\mu\nu}^A=\partial_\mu A_\nu-\partial_\nu A_\mu+[V_\mu,A_\nu]+[A_\mu,V_\nu],$$

and

$$\nabla_\mu H=\partial_\mu H+[V_\mu,H]-\gamma^5\{A_\mu,H\}$$

is the covariant derivative.

The asymptotic behavior of the operator $\mathbf{A}$ at short distances is determined by its "free" part

$$\mathbf{A}_0=\Box+\mu^2,$$

which corresponds to the nonperturbative part of the heat kernel K_0. The latter can be written in the form

$$\langle x|K_0(\tau)|y\rangle=\langle x|\exp(-(\Box+\mu^2)\tau)|y\rangle$$
$$=\frac{1}{(4\pi\tau)^2}\,e^{-\mu^2\tau+(x-y)^2/(4\tau)}, \tag{3.2}$$

as the solution of the equation

$$\frac{\partial}{\partial\tau}K_0+\mathbf{A}_0K_0=0,\quad K(\tau=0)=1.$$

Using the substitution $K=K_0H$, from the heat kernel we can isolate the interacting part, which satisfies the equation

$$\left(\frac{\partial}{\partial\tau}+\frac{1}{\tau}\,z_\mu d^\mu+d^\mu d_\mu+a\right)H(x,y;\tau)=0,$$

$$H(x,y;\tau=0)=1, \tag{3.3}$$

where $z_\mu=x_\mu-y_\mu$, and the differential operator d_μ acts only on x. The interacting part of the heat kernel can be expanded in powers of the proper time τ:

$$H(x,y;\tau)=\sum_k h_k(x,y)\cdot\tau^k.$$

The resulting coefficients of the Seeley–DeWitt expansion $h_k(x,y)$ satisfy the recursion relation

$$(n+z_\mu d^\mu)h_{n+1}(x,y)=-(a+d^\mu d_\mu)h_{n-1}(x,y) \tag{3.4}$$

with the boundary condition

$$z_\mu d^\mu h_0=0. \tag{3.5}$$

Therefore, the heat kernel for the modulus of the quark determinant of the bosonized NJL model can be written as an expansion:

$$\langle x|K(\tau)|y\rangle=\frac{1}{(4\pi\tau)^2}\,e^{-\mu^2\tau+(x-y)^2/(4\tau)}\sum_k h_k(x,y)\cdot\tau^k. \tag{3.6}$$

After integration over τ in (3.1) we obtain

$$\ln|\det i\hat{\mathbf{D}}|=-\frac{1}{2}\,\frac{\mu^4}{(4\pi)^2}\sum_k\frac{\Gamma(k-2,\mu^2/\Lambda^2)}{\mu^{2k}}\,\text{Tr}\,h_k, \tag{3.7}$$

where $\Gamma(n,x)=\int_x^\infty dt\,e^{-t}t^{n-1}$ is the incomplete gamma function.

Using the definition of the function $\Gamma(\alpha,x)$ for integer values $\alpha=-n$ and $\alpha=0$, we can separate the divergent and finite parts of the quark determinant (3.7):

$$\frac{1}{2}\ln(\det\hat{\mathbf{D}}^{\dagger}\hat{\mathbf{D}})=B_{\rm pol}+B_{\rm log}+B_{\rm fin}.$$

Here

$$B_{\rm pol}=\frac{1}{2}\frac{e^{-x}}{(4\pi)^2}\left[-\frac{\mu^4}{2x^2}\mathrm{Tr}'\,h_0+\frac{1}{x}\left(\frac{\mu^4}{2}\mathrm{Tr}'\,h_0-\mu^2\,\mathrm{Tr}'\,h_1\right)\right]$$

has a pole at $x=0$ $(x=\mu^2/\Lambda^2)$, and

$$B_{\rm log}=-\frac{1}{2}\frac{1}{(4\pi)^2}\Gamma(0,x)\left[\frac{1}{2}\mu^4\,\mathrm{Tr}'\,h_0-\mu^2\,\mathrm{Tr}'\,h_1+\mathrm{Tr}'\,h_2\right]$$

diverges logarithmically, because

$$\Gamma(0,x)=-(C+\ln x)+O(x),$$

where $C=0.577$ is the Euler constant. The finite part of the modulus of the quark determinant has the form

$$B_{\rm fin}=-\frac{1}{2}\frac{1}{(4\pi)^2}\sum_{k=3}^{\infty}\mu^{4-2k}\Gamma(k-2,x)\mathrm{Tr}'\,h_k.$$

The coefficients of the divergent contributions $\Gamma(-1,\mu^2/\Lambda^2)$ and $\Gamma(0,\mu^2/\Lambda^2)$ correspond to the quadratically and logarithmically divergent one-loop integrals I_1 and I_2 regularized in Ref. 16 by using the momentum cutoff $\bar{\Lambda}=O(1\text{ GeV})$. In contrast to the standard cutoff regularization used in elementary-particle physics, here the cutoff scale is treated as a physical parameter characterizing the $\bar{q}q$ force range and the limit of the region of spontaneous chiral symmetry breaking.

In what follows we shall restrict ourselves to the detailed study of only the nonanomalous part of the effective action associated with the modulus of the quark determinant. The anomalous part of the effective action, which is determined by the complex phase of the quark determinant, can be written as

$$\Gamma^-=\Gamma_{WZ}+\Gamma^-_{\rm (h.o.)}. \tag{3.8}$$

Here $\Gamma_{\rm WZ}$ are the lowest-order anomalous contributions corresponding to order p^4 in the momentum:

$$\Gamma_{WZ}=\frac{iN_c}{240\pi^2}\int_{B_5}d^5x\varepsilon^{\mu\nu\alpha\beta\gamma}\,\mathrm{tr}(\tilde{L}_\mu\tilde{L}_\nu\tilde{L}_\alpha\tilde{L}_\beta\tilde{L}_\gamma)-\frac{iN_c}{48\pi^2}\int d^4x\varepsilon^{\mu\nu\alpha\beta}\,\mathrm{tr}(Z_{\mu\nu\alpha\beta}(U,A_L,A_R)-Z_{\mu\nu\alpha\beta}(1,A_L,A_R)). \tag{3.9}$$

In the first topological term of (3.9) the pseudoscalar chiral fields $\tilde{L}_\mu=\partial_\mu U\cdot U^\dagger$, $U=\Omega^2$, are defined on the disk B_5 in the 5-dimensional integration region bounded by 4-dimensional Euclidean spacetime.[60] This term gives anomalous contributions of the form

$$\frac{iN_c}{240\pi^2}\int_{B_5}d^5x\varepsilon^{\mu\nu\alpha\beta\gamma}\,\mathrm{tr}(\tilde{L}_\mu\tilde{L}_\nu\tilde{L}_\alpha\tilde{L}_\beta\tilde{L}_\gamma)=-\frac{iN_c}{30\sqrt{2}\pi^2F_0^5}\int d^4x\varepsilon^{\mu\nu\alpha\beta}\,\mathrm{tr}(\varphi\partial_\mu\varphi\partial_\nu\varphi\partial_\alpha\varphi\partial_\beta\varphi)+O(\varphi^7). \tag{3.10}$$

The second term in (3.9) describes the anomalous interactions of the pseudoscalar degrees of freedom with the vector and axial-vector gauge fields. The explicit expression for $Z_{\mu\nu\alpha\beta}$ is conveniently written as[61]

$$\begin{aligned}Z_{\mu\nu\alpha\beta}(U,A_L,A_R)=&A^U_{L\mu}(A_{R\nu}\partial_\alpha A_{R\beta}+\partial_\nu A_{R\alpha}A_{R\beta}\\&+A_{R\nu}A_{R\alpha}A_{R\beta}-R_\nu R_\alpha A_{R\beta})\\&+U^\dagger A_{L\mu}U(A_{R\nu}R_\alpha A_{R\beta}-R_\nu\partial_\alpha A_{R\beta})\\&+\frac{1}{2}A_{L\mu}L_\nu A_{L\alpha}L_\beta-(A_L\leftrightarrow A_R)\\&+\frac{1}{2}(A_{L\mu}UA^\dagger_{R\nu})(A_{L\alpha}UA_{R\beta}U^\dagger),\end{aligned}$$

where

$$A^U_{L\mu}=U^\dagger A_{L\mu}U+R_\mu,\quad A^U_{R\mu}=UA_{R\mu}U^\dagger-L_\mu.$$

The contributions additional to $\Gamma_{\rm WZ}$ of higher order in the momenta $\Gamma^-_{\rm (h.o.)}$ in the full anomalous action (3.8) can be obtained in the approach of Ref. 62 by using the same heat-kernel coefficients h_n as those arising in the calculation of the modulus of the quark determinant:

$$\Gamma^-_{\rm (h.o.)}=-i\int d^4x\sum_{n=3}^{\infty}\sum_{r=1}^{\infty}\frac{(n-2)!}{16\pi^2r}\left[\frac{1}{\mu^{2n-2}}\mathrm{tr}((P_R\Phi^\dagger+P_L\Phi)\hat{\mathbf{D}}h_n)\right]_{r\varepsilon}. \tag{3.11}$$

The subscripts ε and r on the square brackets in (3.11) indicate that when calculating the quantity inside the brackets, only terms with an odd number of Levi-Civita tensors and terms of order μ^{-r} are kept.

4. OBTAINING THE HEAT-KERNEL COEFFICIENTS

The recursive algorithm for calculating the heat-kernel coefficients is based on Eq. (3.4) relating $h_n(x,y)$ and $h_{n-1}(x,y)$. The boundary condition (3.5) for $h_0(x,y)$ arises from (3.4) for $n=0$ and $h_{-1}(x,y)=0$. The problem is to find the limit $h_n\equiv h_n(x,y)|_{z=0}$. We cannot simply take $z=0$ in (3.4), because the action of the differential operator d_α on this relation gives rise to a nonvanishing contribution $d_\alpha(z_\mu d^\mu h_n)|_{z=0}=g_{\alpha\mu}d^\mu h_n|_{z=0}=d_\alpha h_n|_{z=0}$.

It is easily seen that the use of the recursion relation (3.4) for calculating the coefficient h_n leads to the appearance of terms of the form $d_\alpha d_\beta\dots h_n(x,y)|_{z=0}$. In order to obtain a recursion relation for such terms, it is necessary to act on Eq. (3.4) with the product of m differential operators

$d_\alpha d_\beta \dots d_\omega$ and take the limit $z=0$:

$$\underbrace{d_\alpha d_\beta \dots d_\omega}_{m} h_n(x,y)|_{z=0} = -\frac{1}{n+m}\{d_\alpha d_\beta \dots d_\omega (a + d_\mu d^\mu) h_{n-1}(x,y) + P_{\alpha\beta\dots\omega} h_n(x,y)\}|_{z=0}, \tag{4.1}$$

where $(n+m)>0$ and

$$P_{\alpha\beta\dots\omega} = \underbrace{d_\alpha d_\beta \dots d_\omega}_{m} \cdot z_\mu d^\mu |_{z=0} - m d_\alpha d_\beta \dots d_\omega .$$

Therefore, for $P_{\alpha\beta\dots\omega}$ we obtain the recursion relation

$$P_{\alpha,\beta\dots\omega} = d_\alpha P_{\beta\dots\omega} + R_{\beta\dots\omega;\alpha} \tag{4.2}$$

with the boundary condition $P=0$, where $R_{\beta\dots\omega;\alpha} = [d_\beta \dots d_\omega, d_\alpha]$. Moving the differential operator d_α step by step through the sequence of other differential operators $d_\beta, \dots, d_\omega$, we end up with it on the left. In this case the two products of m differential operators cancel and only the terms with $(m-2)$ differentials remain. Finally, we obtain the recursion relation

$$R_{\beta\gamma\dots\omega;\alpha} = \Gamma_{\beta\alpha} \cdot d_\gamma \dots d_\omega + d_\beta \cdot R_{\gamma\dots\omega;\alpha} \tag{4.3}$$

with the boundary condition $R_{;\alpha}=0$. Thus, $h_n(x,y)|_{z=0}$ can be calculated using Eq. (4.1) beginning with $m=0$. After each iteration it is necessary to move all the differential operators arising from $d_\mu d^\mu$ or $P_{\alpha\beta\dots\omega}$ to the right to $h_k(x,y)$. This gives rise to commutators of the form

$$S_\mu = [d_\mu, a]; \quad S_{\mu\nu} = [d_\mu, S_\nu], \quad S_{\alpha\mu\nu} = [d_\alpha, S_{\mu\nu}], \dots$$
$$\Gamma_{\mu\nu} = [d_\mu, d_\nu], \quad K_{\alpha\mu\nu} = [d_\alpha, \Gamma_{\mu\nu}],$$
$$K_{\beta\alpha\mu\nu} = [d_\beta, K_{\alpha\mu\nu}], \dots \tag{4.4}$$

The indices n and m change as follows in these iterations. Either $n \to n-1$, or $m \to m-2$, or $n \to n-1; m \to m+2$. It is easy to show that after $2n$ iterations we are left with only $h_0(x,y)$ without differential operators, and the required result is obtained in the end by substituting the limit $z=0$, using $h_0(x,y)|_{z=0}=1$.

The recursive method described here can be used to make the very awkward calculations of the Seeley–DeWitt coefficients algorithmic for computer algebra systems like FORM or REDUCE after their appropriate extension.[49] The expressions for the heat-kernel coefficients contain a large number of terms coupled together by equivalence transformations following from the physical properties of the total derivatives in the effective action and the Jacobi identities. The resulting equations must therefore be reduced to some minimal basis of linearly independent terms. The problem of reducing the final expressions to the minimal basis and the ambiguities which thus arise are studied in detail in Ref. 50. There the heat-kernel coefficients that we obtained are also compared with the results of other studies.

In the minimal basis that we have chosen, the heat-kernel coefficients $h_1, \dots, h_5$ have the form[1)]

$$h_0 = 1,$$

$$h_1 = -a,$$

$$\mathrm{Tr}\, h_2 = \frac{1}{2}\mathrm{Tr}\left\{a^2 + \frac{1}{6}\Gamma_{\mu\nu}^2\right\},$$

$$\mathrm{Tr}\, h_3 = \frac{1}{6}\mathrm{Tr}\left\{-a^3 + \frac{1}{6}S_\mu^2 - \frac{1}{2}a\Gamma_{\mu\nu}^2 + \frac{1}{10}K_{\nu\nu\mu}^2 - \frac{1}{15}\Gamma_{\mu\nu}\Gamma_{\theta\alpha}\Gamma_{\alpha\mu}\right\},$$

$$\begin{aligned}\mathrm{Tr}\, h_4 = \frac{1}{24}\mathrm{Tr}\Big\{ & a^4 + a^2 S_{\mu\mu} + \frac{4}{5}a^2\Gamma_{\mu\nu}^2 + \frac{1}{5}(a\Gamma_{\mu\nu})^2 \\ & - \frac{2}{5}aS_\mu K_{\nu\nu\mu} + \frac{1}{5}S_{\mu\mu}^2 + \frac{4}{15}a\Gamma_{\mu\nu}\Gamma_{\nu\rho}\Gamma_{\rho\mu} \\ & - \frac{2}{5}aK_{\nu\nu\mu}^2 + \frac{2}{15}S_{\alpha\alpha}\Gamma_{\mu\nu}^2 - \frac{8}{15}S_{\beta\gamma}\Gamma_{\gamma\alpha}\Gamma_{\alpha\beta} \\ & + \frac{17}{210}\Gamma_{\mu\nu}^2\Gamma_{\alpha\beta}^2 + \frac{2}{35}\Gamma_{\mu\nu}\Gamma_{\nu\rho}\Gamma_{\mu\sigma}\Gamma_{\sigma\rho} \\ & + \frac{1}{105}\Gamma_{\mu\nu}\Gamma_{\nu\rho}\Gamma_{\rho\sigma}\Gamma_{\sigma\mu} + \frac{1}{420}\Gamma_{\mu\nu}\Gamma_{\rho\sigma}\Gamma_{\mu\nu}\Gamma_{\rho\sigma} \\ & + \frac{16}{105}K_{\mu\alpha\alpha\nu}\Gamma_{\nu\rho}\Gamma_{\rho\mu} + K_{\alpha\alpha\mu}K_{\beta\beta\nu}\Gamma_{\mu\nu} \\ & + \frac{1}{35}K_{\mu\alpha\alpha\nu}^2\Big\},\end{aligned}$$

$$\begin{aligned}\mathrm{Tr}\, h_5 = \frac{1}{240}\mathrm{Tr}\Big\{ & -a^5 - 2a^3 S_{\mu\mu} - a^2 S_\mu^2 - a^3\Gamma_{\mu\nu}^2 \\ & - \frac{2}{3}a^2\Gamma_{\mu\nu}a\Gamma_{\mu\nu} + \frac{2}{3}a^2 S_\mu K_{\nu\nu\mu} - \frac{2}{3}S_\mu S_\nu a\Gamma_{\mu\nu} \\ & - aS_{\mu\mu}S_{\nu\nu} - \frac{2}{3}S_{\mu\mu}S_\nu^2 - \frac{2}{7}a^2\Gamma_{\mu\nu}\Gamma_{\nu\alpha}\Gamma_{\alpha\mu} \\ & - \frac{8}{21}a\Gamma_{\mu\nu}a\Gamma_{\nu\alpha}\Gamma_{\alpha\mu} + \frac{4}{7}a^2 K_{\mu\mu\nu}K_{\alpha\alpha\nu} \\ & + \frac{3}{7}aK_{\mu\mu\nu}aK_{\alpha\alpha\nu} \\ & - \frac{8}{7}aS_{\mu\mu}\Gamma_{\nu\alpha}^2 + \frac{4}{7}aS_\mu K_{\nu\nu\alpha}\Gamma_{\mu\alpha} \\ & + \frac{8}{7}aS_\mu\Gamma_{\mu\nu}K_{\alpha\alpha\nu} - \frac{4}{21}a\Gamma_{\mu\nu}S_{\alpha\alpha}\Gamma_{\mu\nu} - \frac{11}{21}S_\mu^2\Gamma_{\nu\alpha}^2 \\ & + \frac{20}{21}S_\mu K_{\nu\nu\alpha}a\Gamma_{\mu\alpha} + \frac{2}{21}S_\mu S_\nu\Gamma_{\mu\alpha}\Gamma_{\alpha\nu} \\ & - \frac{10}{21}S_\mu S_\nu\Gamma_{\nu\alpha}\Gamma_{\alpha\mu} + \frac{2}{7}S_\mu\Gamma_{\mu\nu}S_\alpha\Gamma_{\alpha\nu} \\ & + \frac{1}{42}S_\mu\Gamma_{\nu\alpha}S_\mu\Gamma_{\nu\alpha} + \frac{8}{21}S_{\mu\mu}S_\nu K_{\alpha\alpha\nu} \\ & - \frac{4}{21}S_\mu S_\nu K_{\mu\alpha\alpha\nu} + \frac{1}{14}S_{\mu\nu\nu}S_{\mu\alpha\alpha} - \frac{17}{84}a\Gamma_{\mu\nu}^2\Gamma_{\alpha\beta}^2\end{aligned}$$

$$-\frac{1}{21}a\Gamma_{\mu\nu}\Gamma_{\nu\alpha}\Gamma_{\mu\beta}\Gamma_{\beta\alpha}-\frac{1}{21}\alpha\Gamma_{\mu\nu}\Gamma_{\nu\alpha}\Gamma_{\alpha\beta}\Gamma_{\beta\mu}$$

$$-\frac{5}{84}a\Gamma_{\mu\nu}\Gamma^2_{\alpha\beta}\Gamma_{\mu\nu}-\frac{13}{84}a\Gamma_{\mu\nu}\Gamma_{\alpha\beta}\Gamma_{\mu\nu}\Gamma_{\alpha\beta}$$

$$-\frac{5}{21}a\Gamma_{\mu\nu}\Gamma_{\alpha\beta}\Gamma_{\beta\nu}\Gamma_{\alpha\mu}-\frac{2}{21}aK_{\mu\mu\nu}\Gamma_{\nu\alpha}K_{\beta\beta\alpha}$$

$$-\frac{2}{7}a\Gamma_{\mu\nu}K_{\mu\alpha\beta}K_{\nu\alpha\beta}-\frac{4}{21}a\Gamma_{\mu\nu}K_{\nu\alpha\alpha\beta}\Gamma_{\beta\mu}$$

$$-\frac{2}{21}a\Gamma_{\mu\nu}K_{\alpha\alpha\mu}K_{\beta\beta\nu}-\frac{4}{21}S_{\mu\mu}\Gamma_{\nu\alpha}\Gamma_{\alpha\beta}\Gamma_{\beta\nu}$$

$$-\frac{4}{21}S_{\mu\nu}\Gamma_{\mu\alpha}\Gamma_{\beta\nu}\Gamma_{\alpha\beta}+\frac{2}{7}S_{\mu}K_{\nu\nu\mu}\Gamma^2_{\alpha\beta}$$

$$-\frac{2}{7}S_{\mu}K_{\nu\nu\alpha}\Gamma_{\mu\beta}\Gamma_{\beta\alpha}$$

$$+\frac{2}{21}S_{\mu}\Gamma_{\mu\nu}K_{\alpha\alpha\beta}\Gamma_{\beta\nu}-\frac{1}{7}aK_{\mu\nu\nu\alpha}K_{\mu\beta\beta\alpha}$$

$$+\frac{2}{21}aK_{\mu\nu\nu\alpha}K_{\alpha\beta\beta\mu}-\frac{3}{28}S_{\mu\mu\nu\nu}\Gamma^2_{\alpha\beta}$$

$$-\frac{1}{42}S_{\mu\mu}K_{\nu\nu\alpha}K_{\beta\beta\alpha}$$

$$-\frac{2}{7}S_{\mu\mu}K_{\nu\alpha\alpha\beta}\Gamma_{\nu\beta}+\frac{1}{7}S_{\mu}K_{\nu\nu\alpha\alpha\beta}\Gamma_{\mu\beta}$$

$$-\frac{47}{126}\Gamma^2_{\mu\nu}\Gamma_{\alpha\beta}\Gamma_{\beta\gamma}\Gamma_{\gamma\alpha}-\frac{11}{189}\Gamma_{\mu\nu}\Gamma_{\nu\alpha}\Gamma_{\mu\beta}\Gamma_{\alpha\gamma}\Gamma_{\gamma\beta}$$

$$+\frac{1}{63}\Gamma_{\mu\nu}\Gamma_{\nu\alpha}\Gamma_{\mu\beta}\Gamma_{\beta\gamma}\Gamma_{\gamma\alpha}$$

$$+\frac{37}{945}\Gamma_{\mu\nu}\Gamma_{\nu\alpha}\Gamma_{\alpha\beta}\Gamma_{\beta\gamma}\Gamma_{\gamma\mu}$$

$$+\frac{1}{126}\Gamma_{\mu\nu}\Gamma_{\nu\alpha}\Gamma_{\beta\gamma}\Gamma_{\alpha\mu}\Gamma_{\beta\gamma}$$

$$+\frac{1}{945}\Gamma_{\mu\nu}\Gamma_{\alpha\beta}\Gamma_{\gamma\mu}\Gamma_{\nu\alpha}\Gamma_{\beta\gamma}$$

$$-\frac{8}{189}K_{\mu\mu\nu}K_{\nu\alpha\beta}\Gamma_{\beta\gamma}\Gamma_{\gamma\alpha}-\frac{10}{189}K_{\mu\mu\nu}K_{\alpha\alpha\nu}\Gamma^2_{\beta\gamma}$$

$$+\frac{2}{21}K_{\mu\mu\nu}K_{\alpha\alpha\beta}\Gamma_{\nu\gamma}\Gamma_{\gamma\beta}+\frac{4}{63}K_{\mu\mu\nu}\Gamma_{\nu\alpha}K_{\beta\beta\gamma}\Gamma_{\gamma\alpha}$$

$$+\frac{5}{378}K_{\mu\mu\nu}\Gamma_{\alpha\beta}K_{\gamma\gamma\nu}\Gamma_{\alpha\beta}-\frac{61}{189}K_{\mu\nu\nu\alpha}\Gamma_{\mu\alpha}\Gamma^2_{\beta\gamma}$$

$$+\frac{22}{189}K_{\mu\nu\nu\alpha}\Gamma_{\mu\beta}\Gamma_{\gamma\alpha}\Gamma_{\beta\gamma}-\frac{16}{189}K_{\mu\nu\nu\alpha}\Gamma_{\alpha\beta}\Gamma_{\beta\gamma}\Gamma_{\gamma\mu}$$

$$-\frac{10}{189}K_{\mu\nu\nu\alpha}\Gamma_{\alpha\beta}\Gamma_{\gamma\mu}\Gamma_{\beta\gamma}-\frac{2}{189}K_{\mu\nu\nu\alpha}\Gamma_{\beta\gamma}\Gamma_{\mu\alpha}\Gamma_{\beta\gamma}$$

$$-\frac{4}{63}K^2_{\mu\nu\alpha}\Gamma^2_{\beta\gamma}-\frac{4}{189}K_{\mu\nu\alpha}K_{\mu\alpha\beta}\Gamma_{\nu\gamma}\Gamma_{\gamma\beta}$$

$$-\frac{4}{189}K_{\mu\nu\alpha}K_{\mu\alpha\beta}\Gamma_{\beta\gamma}\Gamma_{\gamma\nu}-\frac{2}{63}K_{\mu\nu\alpha}K_{\beta\nu\alpha}\Gamma_{\mu\gamma}\Gamma_{\gamma\beta}$$

$$+\frac{2}{63}K_{\mu\nu\alpha}\Gamma_{\nu\alpha}K_{\mu\beta\gamma}\Gamma_{\beta\gamma}+\frac{2}{189}K_{\mu\nu\alpha}\Gamma_{\alpha\beta}K_{\mu\nu\gamma}\Gamma_{\gamma\beta}$$

$$-\frac{1}{42}K_{\mu\mu\nu\nu\alpha}K_{\alpha\beta\gamma}\Gamma_{\beta\gamma}+\frac{1}{63}K_{\mu\nu\nu\alpha}K_{\mu\beta\beta\gamma}\Gamma_{\alpha\gamma}$$

$$+\frac{4}{63}K_{\mu\nu\nu\alpha}K_{\alpha\beta\beta\gamma}\Gamma_{\mu\gamma}-\frac{5}{63}K_{\mu\nu\nu\alpha}K_{\beta\beta\mu}K_{\gamma\gamma\alpha}$$

$$\left.+\frac{1}{126}K_{\mu\mu\nu\nu\alpha}K_{\beta\beta\gamma\gamma\alpha}\right\}+\text{h.c.}$$

Here the operation of Hermitian conjugation (h.c.) is defined as

$$a^\dagger=a,\quad (S_\mu\ldots)^\dagger=S_{\mu\ldots},$$

$$(\Gamma_{\mu\nu})^\dagger=-\Gamma_{\mu\nu},\quad (K_{\alpha\mu\nu\ldots})^\dagger=-K_{\alpha\mu\nu\ldots}.$$

Our expressions for the heat-kernel coefficients $h_{4,5}$ turn out to be equivalent to the results of Refs. 51 and 52, where minimal bases different from ours were used. Meanwhile, in $\mathrm{Tr}\,h_4$ we have found some discrepancies between our results[50] and those of Ref. 53 in the part containing terms with six and eight indices:

$$(2\,\mathrm{Tr}\,h_4)_{[50]}-(24\,\mathrm{Tr}\,h_4)_{[53]}$$

$$=-\frac{8}{15}ad^2_\mu d_\nu d^2_\alpha d_\nu+\frac{8}{15}ad^2_\mu d_\nu d_\alpha d_\nu d_\alpha$$

$$-\frac{8}{15}ad_\mu d^2_\nu d_\mu d^2_\alpha+\frac{8}{15}ad_\mu d_\nu d_\mu d_\nu d^2_\alpha$$

$$+\frac{16}{15}ad_\mu d_\nu d^2_\alpha d_\nu d_\mu-\frac{16}{15}ad_\mu d_\nu d_\alpha d_\nu d_\alpha d_\mu$$

$$-\frac{16}{105}d^2_\mu d^2_\nu d^2_\alpha d^2_\beta-\frac{16}{105}d^2_\mu d^2_\nu d_\alpha d^2_\beta d_\alpha$$

$$+\frac{64}{105}d^2_\mu d^2_\nu d_\alpha d_\beta d_\alpha d_\beta+\frac{32}{105}d^2_\mu d_\nu d_\alpha d_\nu d^2_\beta d_\alpha$$

$$-\frac{64}{105}d^2_\mu d_\nu d_\alpha d_\nu d_\beta d_\alpha d_\beta$$

$$+\frac{16}{105}d^2_\mu d_\nu d_\alpha d^2_\beta d_\nu d_\alpha+\frac{32}{105}d^2_\mu d_\nu d_\alpha d_\beta d_\nu d_\alpha d_\beta$$

$$-\frac{16}{35}d^2_\mu d_\nu d_\alpha d_\beta d_\nu d_\beta d_\alpha$$

$$-\frac{16}{35}d^2_\mu d_\nu d_\alpha d_\beta d_\alpha d_\nu d_\beta+\frac{16}{105}d^2_\mu d_\nu d_\alpha d_\beta d_\alpha d_\beta d_\nu$$

$$-\frac{32}{105}d_\mu d_\nu d_\mu d_\nu d_\alpha d_\beta d_\alpha d_\beta$$

$$+\frac{64}{105}d_\mu d_\nu d_\mu d_\alpha d_\nu d_\beta d_\alpha d_\beta$$

$$-\frac{32}{105}d_\mu d_\nu d_\mu d_\alpha d_\beta d_\nu d_\alpha d_\beta$$

$$+\frac{32}{105}d_\mu d_\nu d_\mu d_\alpha d_\beta d_\nu d_\beta d_\alpha .$$

These discrepancies arise from the terms of the corresponding expression of Ref. 53:

$$\left(+\frac{2}{5}S^\alpha{}_\alpha\Gamma^2_{\mu\nu}+\frac{4}{105}\{\Gamma_{\alpha\beta}|K^\rho{}_{\rho\mu}|K^{\mu\alpha\beta}\}\right),$$

where $\{A|B|C\}\equiv ABC+CBA$. The corresponding part of our expression is given by the terms

$$\left(+\frac{2}{15}S^\alpha{}_\alpha\Gamma^2_{\mu\nu}+\frac{16}{105}K^{\mu\alpha}{}_{\alpha\nu}\Gamma^{\nu\rho}\Gamma_{\rho\mu}\right).$$

To check our results, we also verified that the coefficients that we found satisfy the equation[53]

$$\partial(\mathrm{Tr}\,h_n)/\partial a=-h_{n-1}.$$

Let us also give the "minimal" parts of the coefficients h_5, h_6, and h_7 corresponding to terms nonvanishing for $V_\mu=A_\mu=0$:

$$\mathrm{Tr}\,h_5^{\min}=\frac{1}{120}\mathrm{Tr}\left\{-a^5+3a^2S^2_\mu+2aS_\mu aS_\mu-aS^2_{\mu\nu}\right.$$
$$\left.-\frac{5}{3}S_\mu S_\nu S_{\mu\nu}+\frac{1}{14}S^2_{\mu\nu\alpha}\right\},$$

$$\mathrm{Tr}\,h_6^{\min}=\frac{1}{720}\mathrm{Tr}\left\{a^6-4a^3S^2_\mu-6a^2S_\mu aS_\mu+\frac{12}{7}a^2S^2_{\mu\nu}\right.$$
$$+\frac{9}{7}aS_{\mu\nu}aS_{\mu\nu}+\frac{26}{7}aS_{\mu\nu}S_\mu S_\nu$$
$$+\frac{18}{7}aS_\mu S_{\mu\nu}S_\nu+\frac{26}{7}aS_\mu S_\nu S_{\mu\nu}+\frac{9}{7}S^2_\mu S^2_\nu$$
$$+\frac{17}{14}S_\mu S_\nu S_\mu S_\nu-\frac{3}{7}aS^2_{\mu\nu\alpha}-\frac{11}{21}S_{\mu\nu}S_{\nu\alpha}S_{\mu\alpha}$$
$$\left.-S_\mu S_{\mu\nu\alpha}S_{\nu\alpha}-S_\mu S_{\nu\alpha}S_{\mu\nu\alpha}+\frac{1}{42}S^2_{\mu\nu\alpha\beta}\right\},$$

$$\mathrm{Tr}\,h_7^{\min}=\frac{1}{5040}\mathrm{Tr}\left\{-a^7+5a^4S^2_\mu+8a^3S_\mu aS_\mu\right.$$
$$+\frac{9}{2}a^2S_\mu a^2S_\mu-\frac{5}{2}a^3S^2_{\mu\nu}-\frac{9}{2}a^2S_{\mu\nu}aS_{\mu\nu}$$
$$-6a^2S_{\mu\nu}S_\mu S_\nu-\frac{7}{2}a^2S_\mu S_{\mu\nu}S_\nu$$
$$-6a^2S_\mu S_\nu S_{\mu\nu}-\frac{7}{2}aS^2_\mu S^2_\nu-\frac{11}{2}aS_\mu aS_{\mu\nu}S_\nu$$
$$-\frac{11}{2}aS_\mu aS_\nu S_{\mu\nu}-\frac{17}{2}aS_\mu S_\nu aS_{\mu\nu}$$
$$-\frac{17}{2}aS_\mu S_\nu S_\mu S_\nu$$
$$+\frac{5}{6}a^2S^2_{\mu\nu\alpha}+\frac{2}{3}aS_{\mu\nu\alpha}aS_{\mu\nu\alpha}+\frac{17}{6}aS_{\mu\nu\alpha}S_\mu S_{\nu\alpha}$$
$$+\frac{5}{2}aS_{\mu\nu\alpha}S_{\nu\alpha}S_\mu+\frac{5}{3}aS_{\mu\nu}S_{\mu\nu\alpha}S_\alpha$$
$$+\frac{11}{3}aS_{\mu\nu}S_{\nu\alpha}S_{\mu\alpha}$$
$$+\frac{17}{6}aS_{\mu\nu}S_\alpha S_{\mu\nu\alpha}+\frac{5}{3}aS_\mu S_{\mu\nu\alpha}S_{\nu\alpha}$$
$$+\frac{5}{2}aS_\mu S_{\nu\alpha}S_{\mu\nu\alpha}$$
$$+\frac{5}{3}S^2_\mu S^2_{\nu\alpha}+\frac{11}{6}S_\mu S_{\nu\alpha}S_\mu S_{\nu\alpha}+\frac{35}{18}S_\mu S_{\nu\alpha}S_\alpha S_{\mu\nu}$$
$$+\frac{97}{18}S_\mu S_\nu S_{\mu\alpha}S_{\nu\alpha}+\frac{43}{18}S_\mu S_\nu S_{\nu\alpha}S_{\mu\alpha}$$
$$+\frac{35}{9}S_\mu S_\nu S_\alpha S_{\mu\nu\alpha}$$
$$-\frac{1}{6}aS^2_{\mu\nu\alpha\beta}-\frac{16}{15}S_{\mu\nu}S_{\mu\alpha\beta}S_{\nu\alpha\beta}$$
$$-\frac{7}{10}S_{\mu\nu}S_{\alpha\beta}S_{\mu\nu\alpha\beta}-\frac{1}{2}S_\mu S_{\mu\nu\alpha\beta}S_{\nu\alpha\beta}$$
$$\left.-\frac{1}{2}S_\mu S_{\nu\alpha\beta}S_{\mu\nu\alpha\beta}+\frac{1}{132}S^2_{\mu\nu\alpha\beta\gamma}\right\}.$$

Expressions for the heat-kernel coefficients through $\mathrm{Tr}\,h_6^{\min}$ inclusive have also been given in Ref. 56, and an expression for $\mathrm{Tr}\,h_8^{\min}$ can also be found in Ref. 57. Our calculations of the minimal terms of the higher-order heat-kernel coefficients agree with the results of the other groups.

The calculation of the heat-kernel coefficients is of special interest because the use of this method for obtaining an effective Lagrangian from the microscopic theory provides an alternative to the direct calculation of Feynman graphs. Here the ultraviolet divergences of the Feynman graphs will correspond to divergence of the integrals over proper time at the lower limit. An advantage of this method is that the effective Lagrangian can be obtained in the most general form as the expansion of generalized local operators, which will have different forms in different models.

For example, the expressions obtained above for the coefficients h_i describe the effective Lagrangian for the scattering of light on light in quantum electrodynamics, both for particles with spin 1/2 and for spinless particles. In the first case we have $\Gamma_\mu=ieQ\mathscr{A}_\mu$, $a=\frac{1}{4}[\gamma^\mu,\gamma^\nu]\Gamma_{\mu\nu}$, and

$$\mathscr{L}_{\mathrm{eff}}=-\frac{1}{2}\frac{1}{4\pi^2\mu^4}\mathrm{Tr}\,h_4=-\frac{\alpha^2}{180\mu^4}[5(\mathscr{F}^2_{\mu\nu})^2$$
$$-14(\mathscr{F}_{\mu\nu}\mathscr{F}^{\nu\alpha})^2], \qquad (4.5)$$

where $\alpha=e^2/4\pi$ and $\mathscr{F}_{\mu\nu}=\partial_\mu\mathscr{A}_\nu-\partial_\nu\mathscr{A}_\mu$ is the stress tensor of the electromagnetic field. Equation (4.5) coincides with the known Euler–Heisenberg result in QED.[54] In the case of spinless particles we have $\Gamma_\mu=ieQ\mathscr{A}_\mu$, $a=0$, and

$$\mathscr{L}_{\rm eff}=-\frac{1}{2}\frac{1}{4\pi^2\mu^4}\,\mathrm{Tr}\,h_4=-\frac{\alpha^2}{1440\mu^4}[5(\mathscr{F}^2_{\mu\nu})^2+(\mathscr{F}_{\mu\nu}\mathscr{F}^{\nu\alpha})^2].$$

The last expression also coincides with the well known result of scalar electrodynamics.[55] In these simple examples we have used only a few terms from the expression for the coefficient h_4, which in the case of Abelian $U(1)$ symmetry (where all operators commute) are written as

$$\mathrm{Tr}\,h_4\to\frac{1}{24}\,\mathrm{Tr}\left[a^4+a^2\Gamma^2_{\mu\nu}+\frac{1}{12}\Gamma^2_{\mu\nu}\Gamma^2_{\alpha\beta}+\frac{1}{15}(\Gamma_{\mu\nu}\Gamma_{\mu\alpha})^2\right].$$

To obtain the bosonized Lagrangian in the NJL model it is necessary to have more complete expressions, which are also used in other problems, for example, for describing the properties of the electroweak sphaleron,[56] and also for problems in quantum gravity[48] and high-temperature physics. A detailed bibliography of the literature devoted to the various physical applications of the heat-kernel coefficients can be found in Refs. 53 and 58. Therefore, the results that we have obtained and used for these analytic calculations are of much more general interest than just bosonization of the NJL model.

5. NONLINEAR EFFECTIVE MESON LAGRANGIANS

Effective meson Lagrangians in terms of collective fields arise from the quark determinant after calculation of the trace over Dirac matrices in $\mathrm{Tr}\,h_i(x)$. Here we shall restrict ourselves to only terms in the expressions for $h_1,\dots,h_4$ which contribute to the momentum expansion of the quark determinant, including order p^6 [terms contributing in order $O(p^8)$ have been dropped]:

$$h_0=1,$$

$$h_1=-a,$$

$$\mathrm{Tr}\,h_2=\mathrm{Tr}\left\{\frac{1}{12}(\Gamma_{\mu\nu})^2+\frac{1}{2}a^2\right\},$$

$$\mathrm{Tr}\,h_3=-\frac{1}{12}\,\mathrm{Tr}\left\{2a^3-S_\mu S^\mu+a(\Gamma_{\mu\nu})^2-\frac{2}{45}(\Gamma_{\alpha\beta\gamma})^2-\frac{1}{9}(\Gamma^\alpha{}_{\alpha\beta})^2-\frac{2}{45}\Gamma_{\mu\nu}\Gamma^{\nu\alpha}\Gamma_\alpha{}^\mu\right\},$$

$$\mathrm{Tr}\,h_4=\mathrm{Tr}\left\{\frac{1}{24}a^4+\frac{1}{12}(a^2S^\mu{}_\mu+aS_\mu S^\mu)+\frac{1}{720}(7(S^\mu{}_\mu)^2-(S_{\mu\nu})^2)+\frac{1}{30}a^2(\Gamma_{\mu\nu})^2+\frac{1}{120}(a\Gamma_{\mu\nu})^2+\frac{1}{144}a[\Gamma^\mu{}_{\mu\nu},S^\nu]+\frac{1}{40}a\left(\Gamma_{\mu\nu}S^{\mu\nu}+\frac{11}{9}S_{\mu\nu}\Gamma^{\mu\nu}\right)\right\},$$

$$\mathrm{Tr}\,h_5^{\rm min}=-\mathrm{Tr}\left\{\frac{1}{120}a^2(a^3-3S_\mu S^\mu)-\frac{1}{60}(aS_\mu)^2\right\},$$

$$\mathrm{Tr}\,h_6^{\rm min}=\frac{1}{720}\,\mathrm{Tr}\,a^6.$$

The "divergent" part of the effective meson Lagrangian is determined by the coefficients h_0, h_1, and h_2 of the expansion (3.7):

$$\mathscr{L}_{\rm div}=\frac{N_c}{16\pi^2}\,\mathrm{tr}\left\{\Gamma\left(0,\frac{\mu^2}{\Lambda^2}\right)\left[D^\mu(\Phi+m_0)\bar{D}_\mu(\Phi+m_0)^\dagger-\mathscr{M}^2+\frac{1}{6}((F^L_{\mu\nu})^2+(F^R_{\mu\nu})^2)\right]+2\left[\Lambda^2e^{-\mu^2/\Lambda^2}-\mu^2\Gamma\left(0,\frac{\mu^2}{\Lambda^2}\right)\right]\mathscr{M}\right\},\tag{5.1}$$

where $\mathscr{M}=(\Phi+m_0)(\Phi+m_0)^\dagger-\mu^2$ and $F^{R/L}_{\mu\nu}=F^V_{\mu\nu}\pm F^A_{\mu\nu}$. The covariant derivatives D_μ and $\bar{D}_\mu$ are defined as

$$D_\mu*=\partial_\mu*+(A^L_\mu*-*A^R_\mu),$$

$$\bar{D}_\mu*=\partial_\mu*+(A^R_\mu*-*A^L_\mu).$$

Taking $\Sigma\approx\mu$ and therefore $\Phi=\mu\Omega^2\equiv\mu U$, the p^2 part of the meson Lagrangian (5.1) can be written as

$$\mathscr{L}_2=-\frac{F_0^2}{4}\,\mathrm{tr}(L_\mu L^\mu)+\frac{F_0}{4}\,\mathrm{tr}(\chi U^\dagger+U\chi^\dagger),\tag{5.2}$$

where $L_\mu=D_\mu UU^\dagger$. The bare constant F_0 and the meson mass matrix $\chi=\mathrm{diag}(\chi_u^2,\chi_d^2,\dots,\chi_n^2)$ are given by

$$F_0^2=yN_c\mu^2/(4\pi^2),$$

$$\chi_i^2=m_0^i\mu/(G_1F_0^2)=-2m_0^i\langle\bar{q}q\rangle F_0^{-2},\tag{5.3}$$

where $y=\Gamma(0,\mu^2/\Lambda^2)$.

The effective Lagrangian in order p^4 of the chiral expansion arises from the logarithmically divergent part of the quark determinant and from the coefficients h_3 and h_4 contributing to the finite part. Using the properties of the covariant derivatives

$$D_\mu(O_1O_2)=(D_\mu O_1)O_2+O_1(\bar{D}'_\mu O_2)=(D'_\mu O_1)O_2+O_1(D_\mu O_2),$$

$$\bar{D}_\mu(O_1O_2)=(\bar{D}_\mu O_1)O_2+O_1(D'_\mu O_2)=(\bar{D}'_\mu O_1)O_2+O_1(\bar{D}_\mu O_2),$$

$$D'_\mu(O_1O_2)=(D'_\mu O_1)O_2+O_1(D'_\mu O_2)=(D_\mu O_1)O_2+O_1(\bar{D}_\mu O_2),$$

$$\bar{D}'_\mu(O_1O_2)=(\bar{D}'_\mu O_1)O_2+O_1(\bar{D}'_\mu O_2)$$

$$
\begin{aligned}
&=(\bar{D}_\mu O_1)O_2+O_1(D_\mu O_2),\\
&[D_\mu,D_\nu]O=F^L_{\mu\nu}O-OF^R_{\mu\nu},\\
&[\bar{D}_\mu,\bar{D}_\nu]O=F^R_{\mu\nu}O-OF^L_{\mu\nu},\\
&[D'_{\mu\nu},D'_\nu]O=[F^L_{\mu\nu},O],\quad [\bar{D}'_\mu,\bar{D}'_\nu]O=[F^R_{\mu\nu},O],
\end{aligned}
\tag{5.4}
$$

where

$$
D'_\mu * =\partial_\mu * +[A^L_\mu,*],\quad \bar{D}'_\mu * =\partial_\mu * +[A^R_\mu,*],
$$

we can represent the corresponding finite part of the effective Lagrangian as

$$
\begin{aligned}
\mathscr{L}^{(p^4)}_{\text{fin}}=\frac{N_c}{32\pi^2\mu^4}\,\mathrm{tr}\bigg\{&\frac{1}{3}[\mu^2D^2\Phi\bar{D}^2\Phi^\dagger-(D^\mu\Phi\bar{D}_\mu\Phi^\dagger)^2]\\
&+\frac{1}{6}(D_\mu\Phi\bar{D}_\nu\Phi^\dagger)^2-\mu^2[\mathscr{M}D_\mu\Phi\bar{D}^\mu\Phi^\dagger\\
&+\bar{\mathscr{M}}\bar{D}_\mu\Phi^\dagger D_\mu\Phi]+\frac{2}{3}\mu^2[D^\mu\Phi\bar{D}^\nu\Phi^\dagger F^L_{\mu\nu}\\
&+\bar{D}^\mu\Phi^\dagger D^\nu\Phi F^R_{\mu\nu}]+\frac{1}{3}\mu^2F^R_{\mu\nu}\Phi^\dagger F^{L\mu\nu}\Phi\\
&-\frac{1}{6}\mu^4[(F^L_{\mu\nu})^2+(F^R_{\mu\nu})^2]\bigg\},
\end{aligned}
\tag{5.5}
$$

where $\bar{\mathscr{M}}=(\Phi+m_0)^\dagger(\Phi+m_0)-\mu^2$. We shall use the approximation $\Gamma(k,\mu^2/\Lambda^2)\approx\Gamma(k)$, which is satisfied well for $k\geqslant 1$ and $\mu^2/\Lambda^2\ll 1$.

Combining the Lagrangian (5.5) with the p^4 contributions arising from the divergent part (5.1), we obtain the effective p^4 Lagrangian, which in the most general case can be written as

$$
\begin{aligned}
\mathscr{L}_4=\mathscr{L}^{\text{G\&L}}_4(L'_i,H'_i)+\lambda_1\,\mathrm{tr}(D^2U\bar{D}^2U^\dagger)+\lambda_2\,\mathrm{tr}(D^2U\chi^\dagger\\
+\chi\bar{D}^2U^\dagger),
\end{aligned}
\tag{5.6}
$$

where $\mathscr{L}^{\text{G\&L}}_4(L_i,H_i)$ is the part of the p^4 Lagrangian containing the minimal combination of linearly independent terms corresponding to the structure coefficients L_i ($i=1,\dots,10$) and $H_{1,2}$ of Gasser and Leutwyler (Ref. 39):[2)]

$$
\begin{aligned}
\mathscr{L}^{\text{G\&L}}_4(L_i,H_i)=&\left(L_1-\frac{1}{2}L_2\right)(\mathrm{tr}\,L_\mu L^\mu)^2+L_2\,\mathrm{tr}\bigg(\frac{1}{2}[L_\mu,L_\nu]^2\\
&+3(L_\mu L^\mu)^2\bigg)+L_3\,\mathrm{tr}((L_\mu L^\mu)^2)\\
&-L_4\,\mathrm{tr}(L_\mu L^\mu)\mathrm{tr}(\chi U^\dagger+U\chi^\dagger)\\
&-L_5\,\mathrm{tr}[L_\mu L^\mu(\chi U^\dagger+U\chi^\dagger)]+L_6(\mathrm{tr}(\chi U^\dagger\\
&+U\chi^\dagger))^2+L_7(\mathrm{tr}(\chi U^\dagger-U\chi^\dagger))^2\\
&+L_8\,\mathrm{tr}(\chi U^\dagger\chi U^\dagger+U\chi^\dagger U\chi^\dagger)\\
&-L_9\,\mathrm{tr}(F^R_{\mu\nu}R^\mu R^\nu+F^L_{\mu\nu}L^\mu L^\nu)\\
&-L_{10}\,\mathrm{tr}(U^\dagger F^R_{\mu\nu}UF^{L\mu\nu})-H_1\,\mathrm{tr}((F^R_{\mu\nu})^2\\
&+(F^L_{\mu\nu})^2)+H_2\,\mathrm{tr}(\chi\chi^\dagger).
\end{aligned}
\tag{5.7}
$$

The Lagrangian (5.6) contains two extra terms with structure coefficients $\lambda_{1,2}$ which are absent in the standard representation of the p^4 Lagrangian of Gasser and Leutwyler (5.7).

According to the equivalence theorem, a nonlinear effective Lagrangian is defined up to transformations of the pseudoscalar fields which do not change its kinetic part.[64,65] Such field transformations affect only multiparticle propagators with off-shell external lines, while the on-shell S matrices remain unchanged. This equivalence property is used to eliminate extra, for example, tachyonic, terms with double derivatives, which arise in the bosonization of effective quark models. The equations of motion were used for this in Ref. 39. It should be noted that in order p^4 the use of the field transformations leads to exactly the same result as the naive application of the equations of motion arising from the p^2 part of the effective Lagrangian. However, it can be shown that in the next highest order of the momentum expansion, $O(p^6)$, the field transformations lead to the appearance of contributions which can be lost when only the equations of motion are used (see Refs. 37 and 63).

Applying successive covariant differentiation to the unitarity condition $UU^\dagger=1$, we obtain the relations

$$
\begin{aligned}
D^2UU^\dagger&=\frac{1}{2}(D^2UU^\dagger-U\bar{D}^2U^\dagger)-D^\mu U\bar{D}^\mu U^\dagger,\\
U\bar{D}^2U^\dagger&=-\frac{1}{2}(D^2UU^\dagger-U\bar{D}^2U^\dagger)-D^\mu U\bar{D}^\mu U^\dagger.
\end{aligned}
\tag{5.8}
$$

Using (5.8) and $\mathrm{tr}(D^2UU^\dagger-U\bar{D}^2U^\dagger)=0$, the last two terms in (5.6) can be brought to the form most convenient for application of the field transformations:

$$
\begin{aligned}
\mathrm{tr}(D^2U\bar{D}^2U^\dagger)=&\,\mathrm{tr}(D_\mu U\bar{D}^\mu U^\dagger D_\nu U\bar{D}^\nu U^\dagger)+\frac{1}{12}(\mathrm{tr}(\chi U^\dagger\\
&-U\chi^\dagger))^2-\frac{1}{4}\mathrm{tr}(U\chi^\dagger U\chi^\dagger+\chi U^\dagger\chi U^\dagger)\\
&+\frac{1}{2}\mathrm{tr}(\chi\chi^\dagger)-\frac{1}{4}\mathrm{tr}((D^2UU^\dagger\\
&-U\bar{D}^2U^\dagger)\mathscr{O}^{(2)}_{\text{EOM}})-\frac{1}{4}\mathrm{tr}((\chi U^\dagger\\
&-U\chi^\dagger)\mathscr{O}^{(2)}_{\text{EOM}}),\\
\mathrm{tr}(D^2U\chi^\dagger+\chi\bar{D}^2U^\dagger)=&-\mathrm{tr}(D_\mu U\bar{D}^\mu U^\dagger(\chi U^\dagger+U\chi^\dagger))\\
&+\frac{1}{6}(\mathrm{tr}(\chi U^\dagger-U\chi^\dagger))^2\\
&-\frac{1}{2}\mathrm{tr}(U\chi^\dagger U\chi^\dagger+\chi U^\dagger\chi U^\dagger)\\
&+\mathrm{tr}(\chi\chi^\dagger)-\frac{1}{2}\mathrm{tr}((\chi U^\dagger\\
&-U\chi^\dagger)\mathscr{O}^{(2)}_{\text{EOM}}),
\end{aligned}
$$

where the operator $\mathscr{O}^{(2)}_{\text{EOM}}$ has the functional form of the equation of motion arising from the p^2 Lagrangian (5.2):

$$\mathcal{O}^{(2)}_{\rm EOM}(U)=D^2UU^\dagger-U\bar{D}^2U^\dagger-\chi U^\dagger+U\chi^\dagger+\frac{1}{3}\,{\rm tr}(\chi U^\dagger -U\chi^\dagger). \quad (5.9)$$

Then Eq. (5.6) can be written as

$$\mathcal{L}_4=\mathcal{L}_4^{\rm G\&L}(L_i,H_i)+c_1\,{\rm tr}((D^2UU^\dagger-U\bar{D}^2U^\dagger)\mathcal{O}^{(2)}_{\rm EOM}) +c_2\,{\rm tr}((\chi U^\dagger-U\chi^\dagger)\mathcal{O}^{(2)}_{\rm EOM}). \quad (5.10)$$

The primed and unprimed structure coefficients in (5.6) and (5.10) are related to each other as

$$L_1=L_1',\quad L_2=L_2',\quad L_3=L_3'+\lambda_1,\quad L_4=L_4',$$
$$L_5=L_5'-\lambda_2,$$
$$L_6=L_6',\quad L_7=L_7'+\frac{\lambda_1}{12}+\frac{\lambda_2}{6},$$
$$L_8=L_8'-\frac{\lambda_1}{4}-\frac{\lambda_2}{2},\quad L_9=L_9',$$
$$L_{10}=L_{10}',\quad H_1=H_1',\quad H_2=H_2'+\frac{\lambda_1}{2}+\lambda_2,$$
$$c_1=-\frac{\lambda_1}{4},\quad c_2=-\frac{\lambda_1}{4}-\frac{\lambda_2}{2}.$$

In the case of the bosonization of the NJL model, the coefficients $\lambda_{1,2}$ are given by $\lambda_1=\frac{1}{6}N_c/16\pi^2$ and $\lambda_2=-(N_c/16\pi^2)xy$.

The field-transformation method can be used to get rid of the last two terms in the Lagrangian (5.10). For this we consider the transformation

$$U(x)=\exp(iS_2(V))V(x),$$

where $S_2(V)$ is the most general form of the generator in order p^2 of the momentum expansion:

$$S_2(V)=i\alpha_1(D^2VV^\dagger-V\bar{D}^2V^\dagger)+i\alpha_2\Big(\chi V^\dagger-V\chi^\dagger -\frac{1}{3}\,{\rm tr}(\chi V^\dagger-V\chi^\dagger)\Big),$$

with arbitary real parameters α_1 and α_2. Let us consider how the functional form of the Lagrangian $\mathcal{L}_2$ (5.2) changes under field transformations.[3] Substituting $U=\exp(iS)V$ into $\mathcal{L}_2(U)$ and discarding the total derivatives, we obtain

$$\mathcal{L}_2(U)=\mathcal{L}_2(V)+\delta^{(1)}\mathcal{L}_2(V,S)+\delta^{(2)}\mathcal{L}_2(V,S)+\ldots, \quad (5.11)$$

where the superscripts in parentheses denote the power of S (or $D_\mu S,\ldots$) and the corresponding contributions can be written as

$$\delta^{(1)}\mathcal{L}_2(V,S)=\frac{F_0^2}{4}\,{\rm tr}(iS\mathcal{O}^{(2)}_{\rm EOM}(V))=O(p^4),$$

$$\delta^{(2)}\mathcal{L}_2(V,S)=\frac{F_0^2}{4}\,{\rm tr}\Big(S(D_\mu VV^\dagger D'^\mu S-D'^\mu SD_\mu VV^\dagger -D'^2S)-\frac{1}{2}(\chi V^\dagger+V\chi^\dagger)S^2\Big)=O(p^6),$$

$$\delta^{(3)}\mathcal{L}_2(V,S)=O(p^2)\times O(S^3)=O(p^8).$$

The last term is interesting only when $O(p^8)$ contributions are included, and so we do not give its explicit form here.

In what follows we shall not assume that Eq. (5.9) is zero, in spite of the fact that it has the functional form of the equations of motion derived from $\mathcal{L}_2$. If

$$\alpha_1=\frac{4c_1}{F_0^2}=-\frac{\lambda_1}{F_0^2},\quad \alpha_2=\frac{4c_2}{F_0^2}=-\frac{\lambda_1}{F_0^2}-\frac{2\lambda_2}{F_0^2},$$

the term $\delta^{(1)}\mathcal{L}_2(V,S_2)$ exactly cancels the last two terms in (5.10) ($U\to V$ in order p^4), and the effective p^4 Lagrangian leads to the minimal form (5.7) with structure constants $L_i=N_c/(16\pi^2)\,l_i$ and $H_i=N_c/(16\pi^2)h_i$, given by the expressions

$$l_1=\frac{1}{24},\quad l_2=\frac{1}{12},\quad l_3=-\frac{1}{6},\quad l_4=0,$$
$$l_5=x(y-1),\quad l_6=0,$$
$$l_7=-\frac{1}{6}\Big(xy-\frac{1}{12}\Big),\quad l_8=\Big(\frac{1}{2}x-x^2\Big)y-\frac{1}{24},$$
$$l_9=\frac{1}{3},\quad l_{10}=-\frac{1}{6},$$
$$h_1=-\frac{1}{6}\Big(y-\frac{1}{2}\Big),\quad h_2=-(x+2x^2)y+\frac{1}{12}-2x^2y\Big(1 -\frac{\Lambda^2}{\mu^2y}\,e^{-\mu^2/\Lambda^2}\Big), \quad (5.12)$$

where

$$x=-\mu F_0^2/(2\langle\bar{q}q\rangle),\quad y=4\pi^2F_0^2/(N_c\mu^2). \quad (5.13)$$

The structure coefficients (5.12) agree with the results obtained in Ref. 20.

Let us now consider the effective Lagrangian obtained from bosonization of the NJL model in order p^6. First of all, this Lagrangian receives contributions from the additional terms arising from $\mathcal{L}^{(p^4)}_{\rm fin}$ (5.5) after the substitution[4] $\Phi\to\Phi+m_0$, and also the finite part of the effective action $\mathcal{L}^{(p^4)}_{\rm fin}$ corresponding to the heat-kernel coefficients h_3, h_4, h_5, and h_6 [see Eq. (A.1) of the Appendix]. Contributions of order p^6 also arise from $\delta^{(2)}\mathcal{L}_2(V,S_2)$ (5.12). To represent these contributions explicitly it is convenient to introduce the operators

$$A_1={\rm tr}((\chi U^\dagger-U\chi^\dagger)\{D_\mu UU^\dagger,(D^\mu\chi U^\dagger+UD^\mu\chi^\dagger)\}),$$
$$A_2={\rm tr}(\chi U^\dagger-U\chi^\dagger){\rm tr}(D_\mu UU^\dagger(D^\mu\chi U^\dagger+UD^\mu\chi^\dagger)),$$
$$A_3={\rm tr}((\chi U^\dagger-U\chi^\dagger)D_\mu UU^\dagger(\chi U^\dagger-U\chi^\dagger)D^\mu UU^\dagger),$$
$$A_4={\rm tr}(\chi U^\dagger-U\chi^\dagger){\rm tr}(D_\mu UD^\mu U^\dagger(\chi U^\dagger-U\chi^\dagger)),$$
$$A_5={\rm tr}((\chi U^\dagger-U\chi^\dagger)(D^2\chi U^\dagger-UD^2\chi^\dagger)),$$
$$A_6={\rm tr}(\chi U^\dagger-U\chi^\dagger){\rm tr}(D^2\chi U^\dagger-UD^2\chi^\dagger),$$

in terms of which the changes of second order in S_2 are written as

$$(\alpha_1+\alpha_2)^2 \frac{F_0^2}{4}\left(-A_1+\frac{2}{3}A_2+A_3+\frac{1}{3}A_4+A_5-\frac{1}{3}A_6\right). \tag{5.14}$$

In obtaining Eq. (5.14) we discarded the total derivatives and used field transformations with generators of order p^4. (Here we have again used the symbol U in writing down the final expressions after the field transformations. The matrix U now contains not the original, but the transformed interpolating fields.)

Another source of p^6 contributions to the bosonized Lagrangian is the modification of $\mathscr{L}_4$ under field transformations, which can be written as

$$\mathscr{L}_4(U)=\mathscr{L}_4(V)+\delta^{(1)}\mathscr{L}_4(V,S)+O(p^8), \tag{5.15}$$

where

$$\delta^{(1)}\mathscr{L}_4(V,S)=\frac{F_0^2}{4}\operatorname{tr}(iS\mathscr{O}^{(4)}_{\mathrm{EOM}}(V))=O(p^6).$$

From the Lagrangian (5.6) we can obtain the p^4 contribution to the equation-of-motion operator $\mathscr{O}^{(4)}_{\mathrm{EOM}}(U)$:

$$\mathscr{O}^{(4)}_{\mathrm{EOM}}(U)=\frac{4}{F_0}\left(E_4-\frac{1}{3}\operatorname{tr}(E_4)\right),$$

where

$$\begin{aligned}
E_4=&(2L_1'-L_2')\operatorname{tr}(D_\mu U\bar{D}^\mu U^\dagger)\cdot(-U\bar{D}^2U^\dagger+D^2UU^\dagger)\\
&+2L_2'[-U\bar{D}_\mu(\bar{D}_\nu U^\dagger D^\mu U\bar{D}^\nu U^\dagger)\\
&+D_\mu(D_\nu U\bar{D}^\mu U^\dagger D^\nu U)U^\dagger]\\
&+2(2L_2'+L_3')[-U\bar{D}_\mu(\bar{D}^\mu U^\dagger D_\nu U\bar{D}^\nu U^\dagger)\\
&+D_\mu(D_\nu U\bar{D}^\nu U^\dagger D^\mu U)U^\dagger]+L_4'[\operatorname{tr}(\chi U^\dagger+U\chi^\dagger)\\
&\times\times(D^2UU^\dagger-U\bar{D}^2U^\dagger)+\operatorname{tr}(D_\mu U\bar{D}^\mu U^\dagger)\cdot(U\chi^\dagger\\
&-\chi U^\dagger)]\\
&+L_5'[-U\bar{D}_\mu(\bar{D}^\mu U^\dagger(\chi U^\dagger+U\chi^\dagger))\\
&+D_\mu((\chi U^\dagger+U\chi^\dagger)D^\mu U)U^\dagger+U\chi^\dagger D_\mu U\bar{D}^\mu U^\dagger\\
&-D_\mu U\bar{D}^\mu U^\dagger\chi U^\dagger]+2L_6'\operatorname{tr}(\chi U^\dagger+U\chi^\dagger)\cdot(U\chi^\dagger\\
&-\chi U^\dagger)\\
&-2L_7'\operatorname{tr}(\chi U^\dagger-\chi^\dagger)\cdot(U\chi^\dagger+\chi U^\dagger)+L_8'[(U\chi^\dagger)^2\\
&-(\chi U^\dagger)^2]\\
&+L_9'[-U\bar{D}^\nu(F^R_{\mu\nu}\bar{D}^\mu U^\dagger)+D^\mu(D^\nu UF^R_{\mu\nu})U^\dagger\\
&-U\bar{D}^\mu(\bar{D}^\nu U^\dagger F^L_{\mu\nu})+D^\nu(F^L_{\mu\nu}D^\mu U)U^\dagger]\\
&-L_{10}'[UF^R_{\mu\nu}U^\dagger F^{L\mu\nu}-F^L_{\mu\nu}UF^{R\mu\nu}U^\dagger]+\lambda_1'[U\bar{D}^2\bar{D}^2U^\dagger\\
&-D^2D^2UU^\dagger]+\lambda_2'[U\bar{D}^2\chi^\dagger-D^2\chi U^\dagger].
\end{aligned} \tag{5.16}$$

In the end, the effective p^6 Lagrangian can be reduced to a minimal number of linearly independent structures with nonzero coefficients[5)] Q_i:

$$\begin{aligned}
\mathscr{L}_6=\operatorname{tr}\{&Q_1(L_\mu L_\nu L^\nu)^2+Q_2(L_\mu L^\mu)^3+Q_3L_\alpha L^\alpha(L_\mu L_\nu)^2\\
&+Q_4(L_\mu L_\nu L_\alpha)^2+Q_5(L_\mu L_\nu L^\mu)^2\\
&+Q_6(L_\mu L^\mu D_\alpha D_\nu U\bar{D}^\alpha\bar{D}^\nu U^\dagger\\
&+R_\mu R^\mu\bar{D}_\alpha\bar{D}_\nu U^\dagger D^\alpha D^\nu U)\\
&+Q_7(L_\mu L_\nu D_\alpha D^\nu U\bar{D}^\alpha\bar{D}^\mu U^\dagger\\
&+R_\mu R_\nu\bar{D}_\alpha\bar{D}^\nu U^\dagger D^\alpha D^\mu U)\\
&+Q_8(L_\mu L_\nu D_\alpha D^\mu U\bar{D}^\alpha\bar{D}^\nu U^\dagger\\
&+R_\mu R_\nu\bar{D}_\alpha\bar{D}^\mu U^\dagger D^\alpha D^\nu U)+Q_9L_\mu L_\nu L^\nu L^\mu(\chi U^\dagger\\
&+U\chi^\dagger)\\
&+Q_{10}(L_\mu L_\nu)^2(\chi U^\dagger+U\chi^\dagger)+Q_{11}(L_\mu L^\mu)^2(\chi U^\dagger\\
&+U\chi^\dagger)\\
&+Q_{12}(\chi R^\mu U^\dagger(D_\mu D_\nu U+D_\nu D_\mu U)U^\dagger L^\nu\\
&+\chi^\dagger L^\mu U(\bar{D}_\mu\bar{D}_\nu U^\dagger\\
&+\bar{D}_\nu\bar{D}_\mu U^\dagger)UR^\nu)+Q_{13}[\chi(\bar{D}_\mu\bar{D}_\nu U^\dagger L^\mu L^\nu\\
&+R^\nu R^\mu U\bar{D}_\mu\bar{D}_\nu U^\dagger)+\chi^\dagger(D_\mu D_\nu UR^\mu R^\nu\\
&+L^\nu L^\mu D_\mu D_\nu U)]\\
&+Q_{14}[\chi(U^\dagger D_\mu D_\nu U\bar{D}^\mu\bar{D}^\nu U^\dagger+\bar{D}_\mu\bar{D}_\nu U^\dagger D^\mu D^\nu UU^\dagger)\\
&+\chi^\dagger(U\bar{D}_\mu\bar{D}_\nu U^\dagger D^\mu D^\nu U+D_\mu D_\nu U\bar{D}^\mu\bar{D}^\nu U^\dagger U)]\\
&+Q_{15}\chi^\dagger L_\mu\chi R^\mu+Q_{16}(\chi^\dagger\chi R_\mu R^\mu+\chi\chi^\dagger L_\mu L^\mu)\\
&+Q_{17}(U\chi^\dagger U\chi^\dagger L_\mu L^\mu+U^\dagger\chi U^\dagger\chi R_\mu R^\mu)\\
&+Q_{18}[(\chi U^\dagger L_\mu)^2\\
&+(\chi^\dagger UR_\mu)^2]+Q_{19}[(\chi U^\dagger)^3+(\chi^\dagger U)^3]+Q_{20}(U^\dagger\chi\chi^\dagger\chi\\
&+U\chi^\dagger\chi\chi^\dagger)+Q_{21}(F^L_{\mu\nu}\{L_\alpha L^\alpha,L^\mu L^\nu\}\\
&+F^R_{\mu\nu}\{R_\alpha R^\alpha,R^\mu R^\nu\})\\
&+Q_{22}[F^L_{\mu\nu}(L^\mu L_\alpha L^\nu L^\alpha+L_\alpha L^\mu L^\alpha L^\nu)\\
&+F^R_{\mu\nu}(R^\mu R_\alpha R^\nu R^\alpha\\
&+R_\alpha R^\mu R^\alpha R^\nu)]+Q_{23}(F^L_{\mu\nu}L_\alpha L^\mu L^\nu L^\alpha\\
&+F^R_{\mu\nu}R_\alpha R^\mu R^\nu R^\alpha)\\
&+Q_{24}(F^L_{\mu\nu}L^\mu L_\alpha L^\alpha L^\nu+F^R_{\mu\nu}R^\mu R_\alpha R^\alpha R^\nu)\\
&+Q_{25}(F^{L\alpha}{}_{\alpha\nu}L_\mu L^\nu L^\mu-F^{R\alpha}{}_{\alpha\nu}R_\mu R^\nu R^\mu)\\
&+Q_{26}(F^{L\alpha}{}_{\alpha\nu}\{L^\nu,L_\mu L^\mu\}-F^{R\alpha}{}_{\alpha\nu}\{R^\nu,R_\mu R^\mu\})\\
&+Q_{27}[F^L_{\mu\nu}(D^\mu D^\alpha UU^\dagger L_\alpha L^\nu-L^\nu L_\alpha U\bar{D}^\mu\bar{D}^\alpha U^\dagger)\\
&+F^R_{\mu\nu}(\bar{D}^\mu\bar{D}^\alpha U^\dagger UR_\alpha R^\nu\\
&-R^\nu R_\alpha U^\dagger D^\mu D^\alpha U)]+Q_{28}[F^L_{\mu\nu}(L^\nu D^\mu D^\alpha UU^\dagger L_\alpha
\end{aligned}$$

$$
\begin{aligned}
&-L_\alpha U\bar{D}^\mu\bar{D}^\alpha U^\dagger L^\nu)+F^R_{\mu\nu}(R^\nu\bar{D}^\mu\bar{D}^\alpha U^\dagger UR_\alpha\\
&-R_\alpha U^\dagger D^\mu D^\alpha UR^\nu)]\\
&+Q_{29}[(\chi U^\dagger+U\chi^\dagger)\{F^L_{\mu\nu},L^\mu L^\nu\}\\
&+(\chi^\dagger U+U^\dagger\chi)\{F^R_{\mu\nu},R^\mu R^\nu\}]+Q_{30}((\chi U^\dagger-U\chi^\dagger)\\
&\times[F^L_{\mu\nu},L^\mu L^\nu]+(\chi^\dagger U-U^\dagger\chi)\\
&\times[F^R_{\mu\nu},R^\mu R^\nu])+Q_{31}[F^L_{\mu\nu}L^\mu(\chi U^\dagger+U\chi^\dagger)L^\nu\\
&+F^R_{\mu\nu}R^\mu(\chi^\dagger U+U^\dagger\chi)R^\nu]+Q_{32}[\chi(\bar{D}^\mu\bar{D}^\nu U^\dagger F^L_{\mu\nu}\\
&-F^R_{\mu\nu}\bar{D}^\mu\bar{D}^\nu U^\dagger)\\
&+\chi^\dagger(D^\mu D^\nu UF^R_{\mu\nu}-F^L_{\mu\nu}D^\mu D^\nu U)]\\
&+Q_{33}[\chi(U^\dagger F^L_{\mu\nu}D^\mu D^\nu UU^\dagger-U^\dagger D^\mu D^\nu UF^R_{\mu\nu}U^\dagger)\\
&+\chi^\dagger(UF^R_{\mu\nu}\hat{u}D^\mu\bar{D}^\nu U^\dagger U\\
&-U\bar{D}^\mu\bar{D}^\nu U^\dagger F^L_{\mu\nu}U)]+Q_{34}[\chi(U^\dagger F^L_{\mu\nu}U\bar{D}^\mu\bar{D}^\nu U^\dagger\\
&-\bar{D}_\mu\bar{D}_\nu U^\dagger UF^R_{\mu\nu}U^\dagger)+\chi^\dagger(UF^R_{\mu\nu}U^\dagger D^\mu D^\nu U\\
&-D^\mu D^\nu UU^\dagger F^L_{\mu\nu}U)]\\
&+Q_{35}[\chi(U^\dagger D^\mu D^\nu UU^\dagger F^L_{\mu\nu}\\
&-F^R_{\mu\nu}U^\dagger D^\mu D^\nu UU^\dagger)+\chi^\dagger(U\bar{D}^\mu\bar{D}^\nu U^\dagger UF^R_{\mu\nu}\\
&-F^L_{\mu\nu}U\bar{D}^\mu\bar{D}^\nu U^\dagger U)]+Q_{36}[F^{L\alpha}{}_{\alpha\mu}(L^\mu U\chi^\dagger+\chi U^\dagger L^\mu)\\
&-F^{R\alpha}{}_{\alpha\mu}(R^\mu U^\dagger\chi+\chi^\dagger UR^\mu)]+Q_{37}[F^{L\alpha}{}_{\alpha\mu}(L^\mu\chi U^\dagger\\
&+U\chi^\dagger L^\mu)-F^{R\alpha}{}_{\alpha\mu}(R^\mu\chi^\dagger U+U^\dagger\chi R^\mu)]\\
&+Q_{38}[F^{L\alpha}{}_{\alpha\nu}(D^\mu D^\nu UU^\dagger L_\mu\\
&+L_\mu U\bar{D}^\mu\bar{D}^\nu U^\dagger)-F^{R\alpha}{}_{\alpha\nu}(\bar{D}^\mu\bar{D}^\nu U^\dagger UR_\mu\\
&+R_\mu U^\dagger D^\mu D^\nu U)]\\
&+Q_{39}F^L_{\mu\nu}D_\alpha UF^{R\mu\nu}\bar{D}^\alpha U^\dagger+Q_{40}F^L_{\alpha\mu}D_\nu UF^{R\alpha\nu}\bar{D}^\mu U^\dagger\\
&+Q_{41}F^L_{\alpha\mu}D^\mu UF^{R\alpha\nu}\bar{D}_\nu U^\dagger+Q_{42}[(F^L_{\mu\nu})^2L_\alpha L^\alpha\\
&+(F^R_{\mu\nu})^2R_\alpha R^\alpha]\\
&+Q_{43}(L_\alpha L^\alpha UF^R_{\mu\nu}U^\dagger F^{L\mu\nu}\\
&+R_\alpha R^\alpha U^\dagger F^L_{\mu\nu}UF^{R\mu\nu})\\
&+Q_{44}(F^L_{\mu\alpha}F^{L\alpha\nu}L^\mu L_\nu+F^R_{\mu\alpha}F^{R\alpha\nu}R^\mu R_\nu)\\
&+Q_{45}(F^L_{\mu\alpha}F^{L\alpha\nu}L_\nu L^\mu+F^R_{\mu\alpha}F^{R\alpha\nu}R_\nu R^\mu)\\
&+Q_{46}[F^L_{\mu\nu}(D^\mu D_\alpha U+D_\alpha D^\mu U)F^{R\nu\alpha}U^\dagger\\
&+F^R_{\mu\nu}(\bar{D}^\mu\bar{D}_\alpha U^\dagger+\bar{D}_\alpha\bar{D}^\mu U^\dagger)F^{L\nu\alpha}U]\\
&+Q_{47}[F^L_{\mu\nu}F^{L\nu\alpha}(U\bar{D}^\mu\bar{D}_\alpha U^\dagger+D_\alpha D^\mu UU^\dagger)\\
&+F^R_{\mu\nu}F^{R\nu\alpha}(U^\dagger D^\mu D_\alpha U+\bar{D}_\alpha\bar{D}^\mu U^\dagger U)]\\
&+Q_{48}(F^{L\alpha}{}_{\alpha\nu}[L_\mu,UF^{R\mu\nu}U^\dagger-F^{R\alpha}{}_{\alpha\nu}[R_\mu,U^\dagger F^{L\mu\nu}U])\\
&+Q_{49}(F^{L\alpha}{}_{\alpha\nu}[L_\mu,F^{L\mu\nu}]-F^{R\alpha}{}_{\alpha\nu}[R_\mu,F^{R\mu\nu}])\\
&+Q_{50}F^{L\mu}{}_{\mu\alpha}UF^{R\nu\alpha}_{\nu}U^\dagger+Q_{51}[(\chi U^\dagger+U\chi^\dagger)(F^L_{\mu\nu})^2\\
&+(\chi^\dagger U+U^\dagger\chi)(F^R_{\mu\nu})^2]+Q_{52}(\chi U^\dagger F^L_{\mu\nu}UF^{R\mu\nu}U^\dagger\\
&+\chi^\dagger UF^R_{\mu\nu}U^\dagger F^{L\mu\nu}U)\\
&+Q_{53}(\chi F^R_{\mu\nu}U^\dagger F^{L\mu\nu}+\chi^\dagger F^L_{\mu\nu}UF^{R\mu\nu})+Q_{54}[(F^L_{\mu\nu\alpha})^2\\
&+(F^R_{\mu\nu\alpha})^2]+Q_{55}[(F^{L\mu}{}_{\mu\alpha})^2+(F^{R\mu}{}_{\mu\alpha})^2]\\
&+Q_{56}(F^L_{\mu\nu}UF^{R\mu\alpha}U^\dagger F^{L\nu}_\alpha+F^R_{\mu\nu}U^\dagger F^{L\mu\alpha}UF^{R\nu}_\alpha)\\
&+Q_{57}(F^L_{\mu\nu}F^{L\mu\alpha}F^{L\nu}{}_\alpha+F^R_{\mu\nu}F^{R\mu\alpha}F^{R\nu}{}_\alpha)\}\\
&+\mathrm{tr}(\chi U^\dagger-U\chi^\dagger)\mathrm{tr}\{Q_{58}\{L_\mu,L_\nu\}\\
&\times(U\bar{D}_\mu\bar{D}_\nu U^\dagger-D_\mu D_\nu UU^\dagger)+Q_{59}L_\mu L^\mu(\chi U^\dagger-U\chi^\dagger)\\
&+Q_{60}[(\chi U^\dagger)^2-(\chi^\dagger U)^2]+Q_{61}(F^L_{\mu\nu}L^\mu L^\nu-F^R_{\mu\nu}R^\mu R^\nu)\}\\
&+\mathrm{tr}(\chi\bar{D}_\mu U^\dagger-D_\mu U\chi^\dagger)\mathrm{tr}[Q_{62}L_\nu L^\mu L^\nu+Q_{63}(\chi U^\dagger\\
&+U\chi^\dagger)L^\mu\\
&+Q_{64}(F^{L\nu\mu}_\nu-F^{R\nu\mu}_\nu)]+[\mathrm{tr}(\chi U^\dagger-U\chi^\dagger)]^2Q_{65}\,\mathrm{tr}(\chi U^\dagger\\
&+U\chi^\dagger).
\end{aligned}
\qquad (5.17)
$$

Here $Q_i=(N_c/32\pi^2\mu^2)q_i$ and

$$q_1=-\frac{3}{10},\quad q_2=-\frac{7}{30},\quad q_3=\frac{1}{3},\quad q_4=\frac{1}{30},$$

$$q_5=-\frac{1}{10},\quad q_6=\frac{1}{15},$$

$$q_7=\frac{1}{30},\quad q_8=-\frac{1}{30},\quad q_9=\frac{1}{6}(1-2x+2c),$$

$$q_{10}=-\frac{1}{3}\left(\frac{2}{5}+x-c\right),$$

$$q_{11}=\frac{3}{20}-\frac{2}{3}c,\quad q_{12}=\frac{1}{60},$$

$$q_{13}=-\frac{1}{3}\left(\frac{1}{20}-x+c\right),\quad q_{14}=\frac{x}{6},$$

$$q_{15}=\frac{2}{3}x(1-x)-\left(\frac{1}{3}-2x\right)c,$$

$$q_{16}=-\frac{1}{120}+\frac{4}{3}x^2+\frac{x}{6}(1-4x)-2\left(x-\frac{1}{6}\right)c,$$

$$q_{17}=\frac{1}{120}+\frac{x}{6}(1-4x)-\left(x+\frac{1}{6}\right)c,$$

$$q_{18}=\frac{4}{3}x^2+\left(\frac{1}{6}-x\right)c,$$

$$q_{19}=-\frac{1}{240}-x^2+\frac{2}{3}x^3+x(1+2xy)c,$$

$$q_{20}=\frac{1}{240}+x^2+2x^3-4yx^3-x(1+2xy)c,$$

$$q_{21}=-\frac{1}{3},\quad q_{22}=\frac{3}{10},\quad q_{23}=-\frac{1}{6},\quad q_{24}=-\frac{1}{2},$$

$$q_{25}=\frac{2}{15},\quad q_{26}=-\frac{11}{30},$$

$$q_{27}=-\frac{1}{15},$$

$$q_{28}=-\frac{1}{15},\quad q_{29}=\frac{1}{12}\left(\frac{1}{5}-x\right),$$

$$q_{30}=\frac{1}{6}\left(\frac{1}{10}-2x+2c\right),$$

$$q_{31}=\frac{1}{2}\left(\frac{1}{15}-x\right),\quad q_{32}=-\frac{x}{2}+\frac{c}{3},\quad q_{33}=-\frac{x}{6}+\frac{c}{3},$$

$$q_{34}=\frac{x}{6},\quad q_{35}=-\frac{x}{6},\quad q_{36}-\frac{1}{3}\left(\frac{1}{5}-x+c\right),$$

$$q_{37}=\frac{1}{3}\left(\frac{3}{10}-x+c\right),\quad q_{38}=\frac{1}{15},\quad q_{39}=-\frac{1}{180},$$

$$q_{40}=-\frac{1}{5},$$

$$q_{41}=-\frac{1}{5},\quad q_{42}=-\frac{1}{360},\quad q_{43}=\frac{1}{6},$$

$$q_{44}=\frac{1}{3},\quad q_{45}=\frac{1}{5},\quad q_{46}=\frac{1}{6},\quad q_{47}=-\frac{1}{6},$$

$$q_{48}=-\frac{1}{6},\quad q_{49}=-\frac{1}{30},$$

$$q_{50}=\frac{1}{15},$$

$$q_{51}=-\frac{x}{3},\quad q_{52}=-\frac{1}{6}(1+x-c),$$

$$q_{53}=\frac{1}{6}(1+x-c),$$

$$q_{54}=\frac{41}{540},\quad q_{55}=-\frac{7}{135},\quad q_{56}=-\frac{1}{6},\quad q_{57}=\frac{1}{2},$$

$$q_{58}=\frac{1}{180},$$

$$q_{59}=-\frac{1}{30}\left[\frac{1}{4}+5c\left(\frac{1}{3}+2x(1+y)\right)\right],$$

$$q_{60}=-\frac{1}{180}[1+30cx(1+y+4xy)],\quad q_{61}=\frac{1}{45},$$

$$q_{62}=\frac{1}{45},$$

$$q_{63}=\frac{1}{9},\quad q_{64}=\frac{1}{90},\quad q_{65}=\frac{1}{18}\left(\frac{1}{90}-17cxy\right). \tag{5.18}$$

The terms containing the factor $c=(1-1/6y)$ arise from the p^4 part of the equation of motion (5.16); $F^R_{\alpha\mu\nu}=\bar{D}'_\alpha F^R_{\mu\nu}$ and $F^L_{\alpha\mu\nu}=D'_\alpha F^L_{\mu\nu}$.

In obtaining Eq. (5.17) we used the equivalence transformations following from the properties of the covariant derivatives (5.4) and the equations of motion, and also from differentiation of the unitarity condition $UU^\dagger=1$:

$$D_\mu UU^\dagger=-U\bar{D}_\mu U^\dagger,\quad U^\dagger D_\mu U=-\bar{D}_\mu U^\dagger U,$$

$$D_\mu D_\nu UU^\dagger+U\bar{D}_\mu\bar{D}_\nu U^\dagger=-(D_\mu U\bar{D}_\nu U^\dagger+D_\nu U\bar{D}_\mu U^\dagger),$$

$$U^\dagger D_\mu D_\nu U+\bar{D}_\mu\bar{D}_\nu U^\dagger U=-(\bar{D}_\mu U^\dagger D_\nu U+\bar{D}_\nu U^\dagger D_\mu U).$$

6. REDUCTION OF THE VECTOR, AXIAL-VECTOR, AND SCALAR DEGREES OF FREEDOM

In the NJL model studied here, light, composite pseudoscalar Goldstone bosons are the chiral partners of the heavier dynamical vector, axial-vector, and scalar mesons (resonances). Independently of the method of including meson resonances in the effective chiral Lagrangian, the inclusion of resonance exchanges significantly modifies the coupling constants of the low-energy interactions in the pseudoscalar sector. In particular, it was shown in Refs. 67 and 68 that the structure constants L_i of the pseudoscalar p^4 Gasser–Leutwyler Lagrangian are mainly saturated by the contributions of meson-resonance exchange between vertices described by the $O(p^2)$ Lagrangian. Therefore, if the vertices of the p^4 Lagrangian (5.7) and graphs with resonance exchanges between the vertices of the p^2 Lagrangian (5.2) are taken into account simultaneously in calculating the p^4 interaction amplitudes in the pseudoscalar sector, this can lead to the double counting discussed in Ref. 67.

To avoid double counting, in the generating functional (2.5) it is necessary to integrate over the vector, axial-vector, and scalar fields. This reduction of the resonance degrees of freedom gives the effective pseudoscalar Lagrangian with structure constants modified by the contributions of resonance exchange.[69]

To integrate over the vector, axial-vector, and scalar fields in the nonanomalous part of the effective meson action, we shall use the invariance of the modulus of the quark determinant under local $U_L(n)\times U_R(n)$ transformations (2.8).[6)] Using a specially chosen chiral rotation corresponding to the unitary gauge, $\xi_L^\dagger=\xi_R=\Omega$, we can completely eliminate the pseudoscalar degrees of freedom from the rotated Dirac operator. In addition, before performing this rotation it is also convenient to make the shift $\Phi\to\Phi-m_0$, which leads to the appearance of a pseudoscalar mass term from the Gaussian part of the effective action (2.6).[7)]

After these transformations, the rotated Dirac operator takes the form

$$i\hat{\mathbf{D}}\to i\hat{\tilde{\mathbf{D}}}=(P_L\Omega+P_R\Omega^\dagger)i\hat{\mathbf{D}}(P_L\Omega+P_R\Omega^\dagger)=i(\hat{\partial}+\hat{\tilde{V}}+\hat{\tilde{A}}\gamma_5)-\Sigma, \tag{6.1}$$

and the pseudoscalar degrees of freedom are preserved only in the Gaussian part of the effective action quadratic in the fields, which in terms of the rotated fields $\widetilde{V}_\mu$ and $\widetilde{A}_\mu$ (2.8) is written as

$$\widetilde{\mathscr{L}}_0=-\frac{1}{4G_1}\operatorname{tr}[(\mu+m_0+\sigma)^2-(\mu+m_0+\sigma)(\xi_r m_0\xi_L^\dagger+\xi_L m_0\xi_R^\dagger)-m_0^2]-\left(\frac{m_V^0}{g_V^0}\right)^2\operatorname{tr}[(\widetilde{V}_\mu-v_\mu)^2+(\widetilde{A}_\mu-a_\mu)^2]. \tag{6.2}$$

Here the scalar degree of freedom $\sigma(x)$ arises as a quantum fluctuation of the field $\Sigma(x)$ about its vacuum expectation value μ: $\Sigma(x)=\mu+\sigma(x)$; $(m_V^0/g_V^0)^2=1/4G_2$, where m_V^0 and g_V^0 are the bare values of the mass and coupling constant of the vector gauge field; and

$$v_\mu=\frac{1}{2}(\Omega\partial_\mu\Omega^\dagger+\Omega^\dagger\partial_\mu\Omega),$$

$$a_\mu=\frac{1}{2}(\Omega\partial_\mu\Omega^\dagger-\Omega^\dagger\partial_\mu\Omega).$$

The divergent part of the quark determinant for the rotated Dirac operator takes the form

$$\widetilde{\mathscr{L}}_{\text{div}}=\frac{N_c}{16\pi^2}y\operatorname{tr}\Big\{(\partial_\mu\sigma+[\widetilde{V}_\mu,m_0+\sigma])^2-(2\mu\widetilde{A}_\mu+\{\widetilde{A}_\mu,m_0+\sigma\})^2+\frac{1}{6}[(\widetilde{F}_{\mu\nu}^R)^2+(\widetilde{F}_{\mu\nu}^L)^2]-((\mu+m_0+\sigma)^2-\mu^2)^2\Big\}+\frac{N_c}{16\pi^2}2(\Lambda^2 e^{-\mu^2\Lambda^2}-\mu^2 y)\operatorname{tr}[(\mu+m_0+\sigma)^2-\mu^2]. \tag{6.3}$$

If we neglect the quantum fluctuations of the field Σ about its vacuum expectation value, the corresponding p^4 terms of the finite part of the quark determinant take the form

$$\widetilde{\mathscr{L}}_{\text{fin}}^{(4)}(\Sigma=\mu)=\frac{N_c}{32\pi^2}\operatorname{tr}\Big\{[\widetilde{V}_\mu,\widetilde{A}^\mu]^2+\frac{8}{3}(\widetilde{A}_\mu\widetilde{A}_\nu)^2-\frac{8}{3}\widetilde{A}^\mu\widetilde{A}^\nu(\widetilde{F}_{\mu\nu}^L+\widetilde{F}_{\mu\nu}^R)+\frac{1}{3}\widetilde{F}_{\mu\nu}^R\widetilde{F}^{L\mu\nu}-\frac{1}{6}[(\widetilde{F}_{\mu\nu}^L)^2+(\widetilde{F}_{\mu\nu}^R)^2]\Big\}. \tag{6.4}$$

Since the masses of the vector, axial-vector, and scalar mesons are much larger than the pion mass, we can integrate over the scalar field σ and the rotated fields $\widetilde{V}_\mu$ and $\widetilde{A}_\mu$ in the generating functional of the NJL model using the equations of motion arising in the static limit[70] from the mass terms of the Lagrangians (6.2) and (6.3). In this approximation, both the kinetic terms $(\widetilde{F}_{\mu\nu}^{R/L})^2$ for the rotated fields $\widetilde{V}_\mu$ and $\widetilde{A}_\mu$ and the terms with higher-order derivatives are treated as perturbations.

The kinetic terms $(\widetilde{F}_{\mu\nu}^{R/L})^2$ arising from the sum of the Lagrangians (6.3) and (6.4) are reduced to the standard form after renormalization of the rotated unphysical vector and axial-vector fields:

$$\widetilde{V}_\mu=\frac{g_V^0}{(1+\widetilde{\gamma})^{1/2}}\widetilde{V}_\mu^{(\text{ph})},\quad \widetilde{A}_\mu=\frac{g_V^0}{(1-\widetilde{\gamma})^{1/2}}\widetilde{A}_\mu^{(\text{ph})}.$$

Here

$$g_V^0=\left[\frac{N_c}{48\pi^2}(2y-1)\right]^{-1/2},\quad \widetilde{\gamma}=\frac{N_c(g_V^0)^2}{48\pi^2},$$

and $\widetilde{V}_\mu^{(\text{ph})}$ and $\widetilde{A}_\mu^{(\text{ph})}$ are the physical fields of the vector and axial-vector mesons with masses

$$m_\rho^2=\frac{(m_V^0)^2}{1+\widetilde{\gamma}},\quad m_{A_1}^2=Z_A^{-2}\frac{(m_V^0)}{1-\widetilde{\gamma}},$$

where

$$Z_A^2=\left(\frac{m_V^0}{g_V^0}\right)^2\frac{4\pi^2}{N_c\mu^2 y}\left[1+\left(\frac{m_V^0}{g_V^0}\right)^2\frac{4\pi^2}{N_c\mu^2 y}\right]^{-1} \tag{6.5}$$

is a factor taking into account πA_1 mixing.

The static equations of motion arise from variations of the mass terms of the Lagrangians (6.2) and (6.3) with respect to the rotated fields $\widetilde{V}_\mu$, $\widetilde{A}_\mu$, and the scalar field σ. As a result, we find

$$\widetilde{V}_\mu=v_\mu,\quad \widetilde{A}_\mu=Z_A^2 a_\mu,$$

$$\sigma=\frac{Z_A^2}{8x}(\xi_R m_0\xi_L^\dagger+\xi_L m_0\xi_R^\dagger)-m_0, \tag{6.6}$$

To switch on the electromagnetic interaction with the photon field $\mathscr{A}_\mu$, we simply need to make the replacement

$$\widetilde{V}_\mu\to\widetilde{V}_\mu+ie\mathscr{A}_\mu\frac{1}{2}(\xi_R Q\xi_R^\dagger+\xi_L Q\xi_L^\dagger),$$

$$\widetilde{A}_\mu\to\widetilde{A}_\mu+ie\mathscr{A}_\mu\frac{1}{2}(\xi_R Q\xi_R^\dagger-\xi_L Q\xi_L^\dagger). \tag{6.7}$$

To obtain the reduced pseudoscalar Lagrangian taking into account the electromagnetic interaction, we must first make the replacement (6.7) in the quark determinant with the rotated Dirac operator, and then, using the static equations of motion, reconstruct the pseudoscalar degrees of freedom. In the reduction of the vector and axial-vector resonances it is convenient to combine these two steps into the substitutions

$$\widetilde{V}_\mu=v_\mu^{(\gamma)}+ie\mathscr{A}_\mu\frac{1}{2}(\xi_R Q\xi_R^\dagger+\xi_L Q\xi_L^\dagger),$$

$$\widetilde{A}_\mu=Z_A^2\left[a_\mu^{(\gamma)}+ie\mathscr{A}_\mu\frac{1}{2}(\xi_R Q\xi_R^\dagger-\xi_L Q\xi_L^\dagger)\right]$$

or

$$\widetilde{F}_{\mu\nu}^V=(Z_A^4-1)[a_\mu^{(\gamma)},a_\nu^{(\gamma)}]+ie\mathscr{F}_{\mu\nu}\frac{1}{2}(\xi_R Q\xi_R^\dagger+\xi_L Q\xi_L^\dagger), \tag{6.8}$$

$$\widetilde{F}^A_{\mu\nu}=Z_A^2 ie\mathscr{F}_{\mu\nu}\frac{1}{2}(\xi_R Q\xi_R^\dagger-\xi_L Q\xi_L^\dagger). \qquad (6.9)$$

Here

$$v_\mu^{(\gamma)}=\frac{1}{2}(\Omega\partial_\mu^{(\gamma)}\Omega^\dagger+\Omega^\dagger\partial_\mu^{(\gamma)}\Omega),$$

$$a_\mu^{(\gamma)}=\frac{1}{2}(\Omega\partial_\mu^{(\gamma)}\Omega^\dagger-\Omega^\dagger\partial_\mu^{(\gamma)}\Omega)=-\frac{1}{2}\xi_R^\dagger L_\mu^{(\gamma)}\xi_R,$$

$\partial_\mu^{(\gamma)}*=\partial_\mu*+ie_0\mathscr{A}_\mu[Q,*]$ is the extended derivative including bremsstrahlung photon emission, while the stress tensor of the electromagnetic field $\mathscr{F}_{\mu\nu}$ corresponds to a structure photon, and $L_\mu^{(\gamma)}=(\partial_\mu^{(\gamma)}U)U^\dagger$.

Using the equations of motion, we can obtain the kinetic and mass terms of the p^2 Lagrangian in the standard form (5.2) from the terms of the effective Lagrangians (6.2) and (6.3) quadratic in the vector and axial-vector fields. Here the definitions (5.3) for the constant F_0 and the meson matrix χ change as follows after the reduction:[8)]

$$F_0^2=\frac{N_c\mu^2 y}{4\pi^2}\rightarrow F_0^2=Z_A^2\frac{N_c\mu^2 y}{4\pi^2},$$

$$\chi=-2m_0\mu\left(1-\frac{\Lambda^2}{y\mu^2}e^{-\mu^2/\Lambda^2}\right)\rightarrow\chi=\frac{m_0\mu}{G_1F_0^2}. \qquad (6.10)$$

The reduction of the resonance degrees of freedom also leads to modification of the overall structure of the higher-order effective Lagrangians and to redefinition of the corresponding structure coefficients. For example, the reduced Lagrangian describing the strong and electromagnetic interactions in the pseudoscalar sector in order p^4 can be written as

$$\begin{aligned}\mathscr{L}_4^{\rm red}=\mathrm{tr}\Big\{&\frac{1}{2}L_2^{\rm red}[L_\mu^{(\gamma)},L_\nu^{(\gamma)}]^2+(3L_2^{\rm red}+L_3^{\rm red})(L_\mu^{(\gamma)}L^{(\gamma)\mu})^2\\&-L_5^{\rm red}L_\mu^{(\gamma)}L^{(\gamma)\mu}(U\chi^\dagger+\chi U^\dagger)+L_8^{\rm red}(\chi^\dagger U\chi^\dagger U\\&+\chi U^\dagger\chi U^\dagger)-L_9^{\rm red}(ie\mathscr{F}_{\mu\nu})Q(L^{(\gamma)\mu}L^{(\gamma)\nu}\\&+R^{(\gamma)\mu}R^{(\gamma)\nu})-L_{10}^{\rm red}(ie\mathscr{F}_{\mu\nu})^2QUQU^\dagger\\&+H_2^{\rm red}\chi\chi^\dagger\Big\}.\end{aligned}$$

Here $L_i^{\rm red}=(N_c/16\pi^2)\,l_i^{\rm red}$ and $H_i^{\rm red}=(N_c/16\pi^2)\,h_i^{\rm red}$ are the reduced structure coefficients analogous to the coefficients of the standard Gasser–Leutwyler representation (5.7):

$$l_2^{\rm red}=\frac{1}{12}\left[Z_A^8+2(Z_A^4-1)\left(\frac{1}{4}y(Z_A^4-1)-Z_A^4\right)\right],$$

$$l_3^{\rm red}=-\frac{1}{6}\left[Z_A^8+3(Z_A^4-1)\left(\frac{1}{4}y(Z_A^4-1)-Z_A^4\right)\right],$$

$$l_5^{\rm red}=(y-1)\frac{1}{4}Z_A^6,\quad l_8^{\rm red}=\frac{y}{16}Z_A^4,$$

$$l_9^{\rm red}=\frac{1}{3}\left(Z_A^4-\frac{1}{2}y(Z_A^4-1)\right),$$

$$l_{10}^{\rm red}=-\frac{1}{6}(Z_A^4-y(Z_A^4-1)).$$

$$h_2^{\rm red}=yZ_A^2\left(\frac{Z_A^2}{2}-x\right).$$

In this approximation the reduction of scalar resonances contributes only to the coefficients l_5 and l_8, while all the other structure coefficients are saturated by exchanges of vector and axial-vector resonances.

Except for the coefficients $l_3^{\rm red}$ and $l_8^{\rm red}$, our results in the static approximation for the equations of motion agree with the results of Ref. 36 using a different approach for including the contributions of the resonance degrees of freedom. The difference arises from the use in Ref. 36 of operators with derivatives. In our approach such operators correspond to higher-order corrections to the static equations of motion (6.6) arising when in the modulus of the rotated quark determinant we keep also terms linear in the scalar field σ and containing couplings with vector and axial-vector fields and tensor strengths. Such terms arise from both the divergent and the finite part of the quark determinant:

$$\widetilde{\mathscr{L}}_{\rm div}\rightarrow-\frac{N_c}{32\pi^2}y\,\mathrm{tr}(16\mu^2\widetilde{\sigma}\widetilde{A}_\mu^2),$$

$$\begin{aligned}\widetilde{\mathscr{L}}_{\rm fin}^{(4)}\rightarrow\frac{N_c}{32\pi^2}\mathrm{tr}\Big[&16\mu^2\widetilde{\sigma}\widetilde{A}_\mu^2-\frac{4}{3\mu}(\widetilde{F}^L_{\mu\nu}+\widetilde{F}^R_{\mu\nu})\\&\times(\widetilde{A}^\mu\{\widetilde{A}^\nu,\widetilde{\sigma}\}+\{\widetilde{A}^\mu,\widetilde{\sigma}\}\widetilde{A}^\nu)+\frac{1}{3\mu}\widetilde{\sigma}(\widetilde{F}^R_{\mu\nu}\widetilde{F}^{L\mu\nu}\\&+\widetilde{F}^L_{\mu\nu}\widetilde{F}^{R\mu\nu},\end{aligned}$$

$$\widetilde{\mathscr{L}}_{\rm fin}^{(6)}\rightarrow-\frac{N_c}{32\pi^2}\frac{1}{\mu}\mathrm{tr}[\widetilde{\sigma}((\widetilde{F}^L_{\mu\nu})^2+(\widetilde{F}^R_{\mu\nu})^2)],$$

where $\widetilde{\sigma}=\sigma+m_0$. After variation with respect to the fields $\widetilde{V}_\mu$, $\widetilde{A}_\mu$, and σ, we obtain

$$\widetilde{V}_\mu=v_\mu,$$

$$\widetilde{A}_\mu=Z_A^2a_\mu+\frac{2(1-Z_A^2)}{\mu}\frac{1-y}{y}\widetilde{\sigma}\widetilde{A}_\mu,$$

$$\begin{aligned}\sigma=&\frac{Z_A^2}{8x}(\xi_R m_0\xi_L^\dagger+\xi_L m_0\xi_R^\dagger)-m_0-\frac{1}{\mu}\frac{y-1}{y}\widetilde{A}_\mu^2\\&-\frac{1}{12y\mu^3}((\widetilde{F}^V_{\mu\nu})^2+2(\widetilde{F}^A_{\mu\nu})^2). \qquad (6.11)\end{aligned}$$

Substituting Eqs. (6.6) into the left-hand sides of (6.11) as first iterations, we obtain new equations of motion taking into account higher-order corrections to the static approximation. The equation of motion for the vector field is not changed. Use of the new equations of motion leads to the appearance of an additional contribution to the structure coefficient $l_3^{\rm red}$:

$$l_3^{\rm red(h.o.)}=\frac{1}{4}\frac{(y-1)^2}{y}Z_A^8, \qquad (6.12)$$

which, when taken into account, makes our result agree with that of Ref. 36.

Similarly, using the equations of motion (6.11) and the substitutions (6.7), we can obtain the reduced effective Lagrangian describing the strong and electromagnetic interactions in order p^6 of the momentum expansion. Because they are very awkward, here we shall not give the complete expressions for the reduced p^6 Lagrangian and the structure coefficients.

7. THE PHENOMENOLOGY OF LOW-ENERGY MESON PROCESSES IN ORDERS p^4 AND p^6 OF THE CHIRAL THEORY

Let us consider the phenomenology of meson processes at low energies from the viewpoint of verifying the bosonized chiral Lagrangians in order p^4 of the momentum expansion. This topic has been discussed in detail in the literature (see, for example, Refs. 33, 36, 38, 40, and 69 and references therein). The amplitudes of meson processes in order p^4 include the contributions of both Born graphs described by the Lagrangians (5.2) and (5.7), and one-loop graphs with vertices of order p^2 (5.2). The ultraviolet divergences arising from meson loops can be fixed in chiral perturbation theory[39,41] using the corresponding counterterm Lagrangian, or in quantum field theory with meson loops[42] using the superpropagator regularization.[43]

In the standard chiral perturbation theory (Refs. 39 and 41; see also Ref. 40), one uses the fact that in dimensional regularization the divergent part of the one-loop functional has exactly the same form as the Lagrangian (5.7) if the structure coefficients in it are replaced as

$$L_i \to -\Lambda(\tilde{\mu})\Gamma_i, \quad H_i \to -\Lambda(\tilde{\mu})\tilde{\Gamma}_i,$$

$$\Lambda(\tilde{\mu}) = \frac{\tilde{\mu}^{d-4}}{(4\pi)^2}\left\{\frac{1}{d-4} - \frac{1}{2}[\ln 4\pi + 1 + \Gamma'(1)]\right\}.$$

Here Γ_i and $\tilde{\Gamma}_i$ are numerical constants calculated in Ref. 39:

$$\Gamma_1 = \frac{3}{32}, \quad \Gamma_2 = \frac{3}{16}, \quad \Gamma_3 = 0, \quad \Gamma_4 = \frac{1}{8}, \quad \Gamma_5 = \frac{3}{8},$$

$$\Gamma_6 = \frac{11}{144}, \quad \Gamma_7 = 0, \quad \Gamma_8 = \frac{5}{48}, \quad \Gamma_9 = \frac{1}{4},$$

$$\Gamma_{10} = -\frac{1}{4}, \quad \tilde{\Gamma}_1 = -\frac{1}{8}, \quad \tilde{\Gamma}_2 = \frac{5}{24},$$

and $\tilde{\mu}$ is an arbitrary renormalization scale having the dimension of a mass. The ultraviolet divergences in the one-loop functional are eliminated by renormalizing the structure coefficients of the counterterm Lagrangian of the form (5.7). Here renormalized coefficients $L_i^r(\tilde{\mu})$ and $H_i^r(\tilde{\mu})$ are introduced such that all the divergent parts cancel in the one-loop functional:

$$L_i = L_i^r(\tilde{\mu}) + \Gamma_i\Lambda(\tilde{\mu}), \quad H_i = H_i^r(\tilde{\mu}) + \tilde{\Gamma}_i\Lambda(\tilde{\mu}).$$

When this procedure is used, the $\tilde{\mu}$ dependences in the contributions of loops and counterterms cancel each other in the amplitude of any meson process. However, the renormalized coefficients $L_i^r(\tilde{\mu})$ and $H_i^r(\tilde{\mu})$ themselves, being measurable parameters, depend on the choice of renormalization scheme and the scale $\tilde{\mu}$. The latter dependence can be written as

$$L_i^r(\tilde{\mu}_2) = L_i^r(\tilde{\mu}_1) + \frac{\Gamma_i}{(4\pi)^2}\ln\frac{\tilde{\mu}_1}{\tilde{\mu}_2},$$

$$H_i^r(\tilde{\mu}_2) = H_i^r(\tilde{\mu}_1) + \frac{\tilde{\Gamma}_i}{(4\pi)^2}\ln\frac{\tilde{\mu}_1}{\tilde{\mu}_2}. \tag{7.1}$$

From this we see that Γ_i and $\tilde{\Gamma}_i$ are also the coefficients of the so-called chiral logarithms $\sim\ln(p^2/\tilde{\mu}^2)$ in the one-loop functional. The explicit dependence of the measured renormalized coefficients on the arbitrary parameter $\tilde{\mu}$ creates certain difficulties in the phenomenological verification of the bosonized Lagrangians, because it is not at all clear which values of $\tilde{\mu}$ should be used to compare the values of $L_i^r(\tilde{\mu})$ and $H_i^r(\tilde{\mu})$ extracted from the experimental data with the theoretical predictions obtained from bosonization of the NJL model.

This arbitrariness in the choice of the scale $\tilde{\mu}$ can be eliminated by using the results of the superpropagator regularization,[43] which was developed specially for calculating meson loops in quantum field theory with nonlinear effective Lagrangians.[42] The superpropagator approach leads to the same results as the dimensional regularization used in the standard chiral perturbation theory.[39,41] The basic difference is that here the parameter $\tilde{\mu}$ is no longer arbitrary, but is fixed by the natural scale of the chiral expansion, i.e., $\tilde{\mu}_{SP} = 4\pi F_0$. To compare the two approaches, the ultraviolet divergences must be replaced by finite terms in accordance with the substitution

$$(C - 1\varepsilon) \to C_{SP} = 2C + 1 + \frac{1}{2}\left[\frac{d}{dz}(\ln\Gamma^{-2}(2z+2))\right]\Bigg|_{z=0} + \beta\pi = -1 + 4C + \beta\pi,$$

where $\varepsilon = (4-D)/2$, and β is an arbitrary constant arising from the Sommerfeld–Watson integral representation for the superpropagator. Using additional theoretical arguments, including the principle of minimal singularity, it can be shown that the expected values of C_{SP} must lie in the range $C_{SP} \approx 1-4$.

Reasonable estimates for the parameter C_{SP} can be obtained, for example, from the splitting of the decay constants for $K, \pi \to \mu\nu$, using the expressions

$$(F_\pi - F_0)F_0 = 2L_5(\chi_u^2 + \chi_d^2) - \frac{1}{16\pi^2}\frac{2}{3}(m_\pi^2(\tilde{C}_K - 1) + 2m_\pi^2(\tilde{C}_\pi - 1)),$$

$$(F_K - F_0)F_0 = 2L_5(\chi_u^2 + \chi_s^2) - \frac{1}{16\pi^2}\frac{1}{2}(2m_K^2(\tilde{C}_K - 1) + m_\pi^2(\tilde{C}_\pi - 1)), \tag{7.2}$$

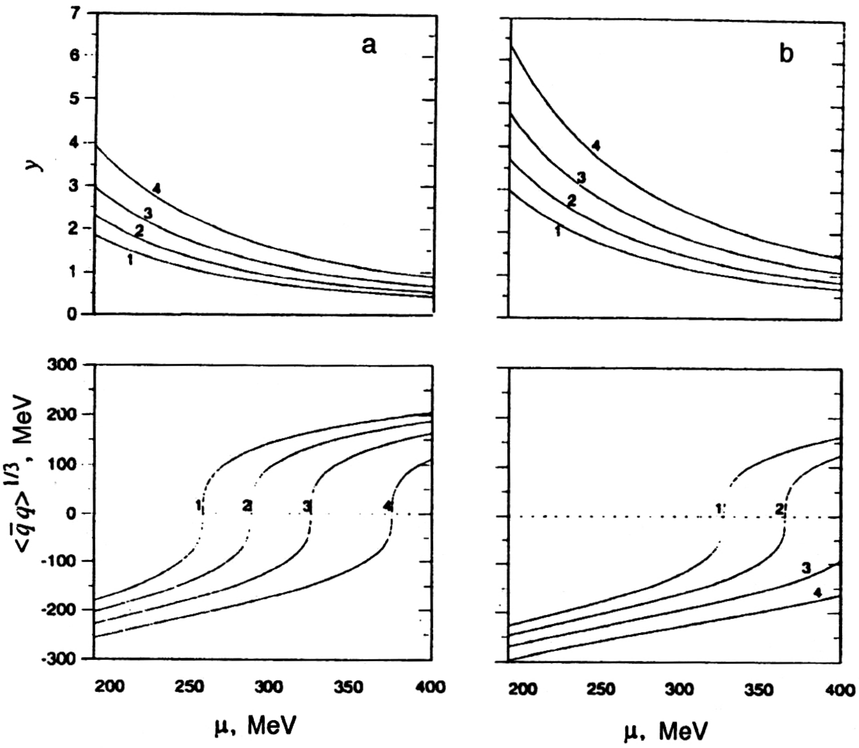

FIG. 3. Dependence of y and $\langle\bar{q}q\rangle^{1/3}$ on the constituent quark mass μ (a) without resonance reduction and (b) after resonance reduction.

where $\tilde{C}_{\pi,K}=C_{SP}+\ln(\pi m^2_{\pi,K}/\mu^2)$. The first terms on the right-hand side of (7.2) proportional to L_5 correspond to the Born contributions,[9] while all the others arise from one-loop graphs of the tadpole type. The splitting of the decay constants F_π and F_K also fixes the parameters x and y related to the quark condensate $\langle\bar{q}q\rangle$ and the averaged constituent quark mass μ by (5.13).

In Fig. 3 we show the dependences of y and $\langle\bar{q}q\rangle^{1/3}$ on the constituent quark mass μ obtained as solutions of (7.2) for fixed values of the parameter C_{SP} and the experimental values of the constants $F_\pi=93$ MeV and $F_K=113$ MeV. The importance of including the reduction of the resonance degrees of freedom in describing the pseudoscalar sector is clearly seen even in this very simple example. The noticeable difference in the behavior of the curves for $\langle\bar{q}q\rangle^{1/3}$ in Fig. 3a (without resonance reduction) and Fig. 3b (after resonance reduction) is related to the appearance of the additional factor of Z_A^2 in the definition (6.10) for the constant F_0. For the numerical estimates we use the phenomenological value of the parameter Z_A^2:

$$Z_A^2=\frac{m_\rho^2}{m_{A_1}^2}\frac{1+\tilde{\gamma}}{1-\tilde{\gamma}}\approx 0.62,$$

which corresponds to $m_\rho=770$ MeV, $m_{A_1}=1260$ MeV, and $g_V=g_{\rho\pi\pi}=6.3$. It should be emphasized that without resonance reduction (Fig. 3a), it is not possible, for any value of C_{SP}, to obtain solutions for y and $\langle\bar{q}q\rangle^{1/3}$ which correspond to the generally accepted ranges of values of the parameters F_0, μ, and m_0. This fixing becomes possible only after resonance reduction (Fig. 3b). For $\mu=265$ MeV it corresponds to the parameter values $y=2.4$, $x=0.10$, $F_0=90$ MeV,

TABLE I. Comparison of the theoretical and phenomenological values of the structure constants $L_i\times 10^3$.*

	Bosonization of the NJL model		Phenomenology	
i	Without resonance reduction	After resonance reduction	$L_i^r(m_\rho)\times 10^3$ (Ref. 44)	$L_i\times 10^3$ (Ref. 33)
1	0.79	0.85	0.4±0.3	0.8±0.2
2	1.58	1.70	1.35±0.3	1.6±0.3
3	−3.17	−4.30	−3.5±1.1	−3.5±0.6
4	0	0	−0.3±0.5	
5	2.66	1.58	1.4±0.5	1.6±0.3
6	0	0	−0.2±0.3	
7	0.50	0	−0.4±0.2	
8	1.03	1.10	0.9±0.3	
9	6.34	7.12	6.9±0.7	6.5±0.5
10	−3.17	−5.90	−5.5±0.7	−3.1±0.7

*The coefficients H_1 and H_2 cannot be fixed phenomenologically.

$\langle\bar{q}q\rangle^{1/3} = -220$ MeV, and $C_{SP} \approx 3$, which is what we shall use in our numerical estimates of the structure constants of bosonized Lagrangians.

In Table I the theoretical predictions of the bosonized NJL model for the constants L_i are compared with the phenomenological values obtained in Ref. 44 using the standard chiral perturbation theory, and also using quantum field theory with superpropagator regularization.[33,69] The most complete analysis of low-energy meson processes performed in Ref. 44 includes a description of the mass spectrum of the pseudoscalar mesons, the splittings of the constants F_π and F_K of the decays $\pi \to e\nu\gamma$ and K_{e4}, and also the Zweig rule. The values of the measured renormalization coefficients $L_i(\tilde{\mu})$ were fixed at $\tilde{\mu} = m_\rho$. The authors of Refs. 33 and 69 used superpropagator regularization to analyze the data on $\pi\pi$ scattering, the charged-pion electromagnetic radius and polarizability, and also the decays $\eta' \to \eta 2\pi$ and $\pi \to e\pi\gamma$. We see from the table that the theoretical predictions for the coefficients L_i are on the whole in good agreement with the phenomenological values. The slight ambiguity in fixing the coefficient L_{10} is related to the fact that in Ref. 44 the data on the form factors of the decay $\pi \to e\nu\gamma$ were used for this, whereas in our analysis[69] we used the data on the process $\gamma\gamma \to \pi^+\pi^-$. These data themselves, especially for the polarizability of the charged pion, apparently need considerable improvement.

The phenomenological verification of the bosonized p^6 Lagrangian is difficult, owing both to the large number of terms in it, and to the fact that in the overwhelming majority of cases the amplitudes of meson processes are dominated by the p^2 and p^4 contributions. Compared to these, the p^6 corrections are so small that it is practically impossible to isolate them from the experimental data. The exceptions are the transitions $\gamma\gamma \to \pi^0\pi^0$ and $\eta \to \pi^0\gamma\gamma$, which turn out to be very sensitive to the p^6 contributions, owing to unique properties of their amplitudes. First of all, in both cases nonzero amplitudes arise only beginning at order p^4 from one-loop graphs, and both ultraviolet divergences and Born contributions are absent in this order. Nonzero Born amplitudes appear only beginning at order p^6. In addition, in the decay $\eta \to \pi^0\gamma\gamma$ pion loops are strongly suppressed by the approximate conservation of G parity, while kaon loops are suppressed by the large mass in the kaon propagator.

The amplitudes of the transition $\gamma\gamma \to \pi^0\pi^0$ in order p^6 with allowance for two-loop meson graphs were first calculated in Ref. 75, using standard chiral perturbation theory, with the structure coefficients of the counterterm Lagrangian in this order fixed from the resonance-exchange model. In Ref. 76 the two transitions $\gamma\gamma \to \pi^0\pi^0$ and $\eta \to \pi^0\gamma\gamma$ were studied by taking into account the Born contributions of the bosonized Lagrangian of the NJL model, with the ultraviolet divergences arising from the one- and two-loop graphs in order p^6 fixed by superpropagator regularization. The role of the reduction of meson resonances in order p^6 was studied in the context of the problem of describing the processes $\gamma\gamma \to \pi^0\pi^0$ and $\eta \to \pi^0\gamma\gamma$ in both Ref. 76 and Ref. 77.

Our calculations[76] show that the resonance reduction in orders p^4 and p^6 significantly improves the description of the experimental data[78] on the total cross sections of the process $\gamma\gamma \to \pi^0\pi^0$ from threshold to the ρ resonance. After reduction of the meson resonances in the bosonized NJL model, we obtained the value 0.11 eV for the width of the decay $\eta \to \pi^0\gamma\gamma$, which does not agree with the experimental value (0.84 ± 0.18) eV. We cannot discuss this problem in greater detail here, and we only note that it is impossible to obtain a quantitative description of the decay $\eta \to \pi^0\gamma\gamma$ in order p^6 of the chiral theory even after the inclusion[79] of the additional contribution arising from resonance exchange with the anomalous $V\pi\gamma$ vertex.[10)]

8. MODIFICATION OF THE NJL MODEL BY NONLOCAL EFFECTS

In Sec. 1 we considered an approximate form of QCD leading to the NJL model in the local limit. The NJL model is based on the assumption that the dominant role in meson dynamics for quark bosonization is played by intermediate momentum transfers $0 \ll q^2 \ll m_G^2$, where the nonperturbative gluon propagator can be approximated as a constant in momentum space and, accordingly, a δ function in coordinate space. In this approach the contribution of the regions of confinement and quark asymptotic freedom is neglected in the effective quark action (1.6). In the region of asymptotic freedom (i.e., for $q^2 \gg m_G^2$) the gluon propagator falls off fairly rapidly with increasing momentum (decreasing distance). Therefore, the exclusion of this region should not lead to any serious distortions of the nonperturbative dynamics of bosonized quarks (mesons). In the extended NJL model that we are considering, the region of asymptotic freedom is excluded by introducing a momentum cutoff at the upper limit corresponding to the parameter Λ arising in the calculation of the quark determinant. In this treatment nonlocal effects arise as contributions from the confinement region.

The description of a wide range of low-energy meson processes using the bosonized NJL model can be viewed both as an indirect phenomenological confirmation that this model is realistic, and as evidence that the chiral dynamics of hadrons is insensitive to quark confinement. However, since the nonperturbative gluon propagator has a pole at zero momentum transfers, discarding the contribution of the confinement region to the effective action integral (1.6) does not at first glance seem as natural a physical approximation as exclusion of the region of quark asymptotic freedom by means of a momentum cutoff. The inclusion of the confinement region leads to the appearance of nonlocal contributions to the effective action (1.6). Therefore, to understand the physical reasons why the hadron chiral dynamics is insensitive to quark confinement, it is sufficient to evaluate the nonlocal corrections to the bosonized effective meson Lagrangian of the NJL model, as has been done in Ref. 23 using a semi-phenomenological bilocal approach.

We begin with the effective action (1.6), which after the introduction of scalar (S), pseudoscalar (P), vector (V), and axial-vector (A) bilocal collective meson fields[1,2,10] can be rewritten in a form bilinear in the quark fields:

$$\mathscr{S}_{\rm int}=\int\int d^4x\, d^4y\Bigg\{-\frac{9}{8D(x-y)}\,{\rm tr}[(\widetilde{S}(x,y))^2$$
$$+(\widetilde{P}(x,y))^2+2((\widetilde{V}_\mu(x,y))^2+(\widetilde{A}_\mu(x,y))^2)]$$
$$+\bar{q}(x)\,\widetilde{\eta}(x,y)q(y)\Bigg\}. \tag{8.1}$$

Here

$$\widetilde{\eta}(x,y)=-\widetilde{S}(x,y)-i\gamma^5\widetilde{P}(x,y)+i\gamma^\mu\widetilde{V}_\mu(x,y)$$
$$+i\gamma^\mu\gamma^5\widetilde{A}_\mu(x,y), \tag{8.2}$$

where

$$\widetilde{S}=\widetilde{S}^a\,\frac{\lambda^a}{2},\quad \widetilde{P}=\widetilde{P}^a\,\frac{\lambda^a}{2},\quad \widetilde{V}_\mu=-i\widetilde{V}^a_\mu\,\frac{\lambda^a}{2},$$

$$\widetilde{A}_\mu=-i\widetilde{A}^a_\mu\,\frac{\lambda^a}{2} \tag{8.3}$$

are collective fields associated with the following bilinear combinations of quarks:

$$\widetilde{S}^a(x,y)=-\frac{8}{9}\,D(x-y)\bar{q}(y)\,\frac{\lambda^a}{2}\,q(x),$$

$$\widetilde{P}^a(x,y)=-\frac{8}{9}\,D(x-y)\bar{q}(y)i\gamma^5\,\frac{\lambda^a}{2}\,q(x),$$

$$\widetilde{V}^a_\mu(x,y)=-\frac{4}{9}\,D(x-y)\bar{q}(y)\gamma_\mu\,\frac{\lambda^a}{2}\,q(x),$$

$$\widetilde{A}^a_\mu(x,y)=-\frac{4}{9}\,D(x-y)\bar{q}(y)\gamma_\mu\gamma^5\,\frac{\lambda^a}{2}\,q(x).$$

In accordance with Ref. 11, assuming strong localization of the bilocal fields, we consider the ansatz

$$\widetilde{\eta}(x,y)\to\widetilde{\eta}(z,t)=\eta(z)f(t)+\eta_\mu(z)t^\mu g(t)+\cdots, \tag{8.4}$$

where $z=(x+y)/2$, $t=(y-x)/2$ are the absolute and relative coordinates, respectively. The function

$$\eta(z)=-S(z)-i\gamma^5P(z)+i\gamma^\mu V_\mu(z)+i\gamma^\mu\gamma^5A_\mu(z) \tag{8.5}$$

is a combination of local collective fields of the composite operators $\bar{q}(z)q(z)$, $\bar{q}(z)i\gamma^5q(z)$, $\bar{q}(z)\gamma_\mu q(z)$, and $\bar{q}(z)\gamma_\mu\gamma^5q(z)$ analogous to the definitions (2.2) and corresponding to the 0^{++}, 0^{-+}, 1^{--}, and 1^{++} low-lying meson states. The term of next order in (8.4) proportional to η_μ can be identified, in the spirit of Ref. 11, with the 1^{--}, 1^{+-}, 2^{++}, and 2^{--} excitations. It is assumed that the bilocal fields $\widetilde{\eta}(x,y)$ are strongly localized on the scale of the effective size of the collective meson $h\equiv1/\widetilde{\Lambda}$, owing to the rapid falloff of the functions $f(t)$ and $g(t)$ for $|t^2|\gg h^2$.

Expanding $q(y)$ and $\bar{q}(x)$ in Taylor series in the region of the absolute coordinate z,

$$q(y)=q(z)+t^\mu\partial_\mu q(z)+O(t^2),$$

$$\bar{q}(x)=q(z)-t^\mu\partial_\mu q(z)+O(t^2),$$

and using (8.4), we obtain

$$\int\int d^4xd^4y\bar{q}(x)\,\widetilde{\eta}(x,y)q(y)$$
$$=2\int d^4z\bar{q}(z)\,\eta(z)q(z)\int d^4tf(t)$$
$$+2\int d^4z\partial^\mu\bar{q}(z)\,\eta(z)\partial_\mu q(z)\int d^4tt^2f(t)$$
$$+(\text{excitation terms}).$$

For the first generation of mesons corresponding to the multiplets 0^{++},0^{-+},1^{--},1^{++}, the generating functional is rewritten as

$$\mathscr{Z}=\int\mathscr{D}\Phi\mathscr{D}\Phi^\dagger\mathscr{D}V\mathscr{D}A\,\exp\Bigg\{\int d^4z$$
$$\Bigg[-\frac{1}{4G_1}\,{\rm tr}[\Phi(z)^\dagger\Phi(z)]-\frac{1}{4G_2}\,{\rm tr}(V^2_\mu(z)+A^2_\mu(z))$$
$$+\bar{q}(z)i\hat{\mathbf{D}}q(z)-\frac{\alpha}{\widetilde{\Lambda}^2}\,\partial^\mu\bar{q}(z)\,\eta(z)\partial_\mu q(z)\Bigg]\Bigg\}, \tag{8.6}$$

where $\hat{\mathbf{D}}$ is the Dirac operator, which coincides with Eq. (2.4) of the extended NJL model. We have introduced the notation $\widetilde{\Lambda}^2=h^{-1}$ in Eq. (8.6), and the parameter α is defined as the integral

$$\frac{\alpha}{\widetilde{\Lambda}^2}=\frac{1}{2}\int d^4t\ t^2f(t), \tag{8.7}$$

where $f(t)$ is normalized as $2\int d^4tf(t)=1$.

The coupling constants G_1 and G_2 are defined as

$$\frac{1}{G_1}=\frac{1}{2G_2}=\frac{9}{8}\int d^4t\,\frac{f^2(t)}{D(2t)}. \tag{8.8}$$

We note that in this approximation the ratio of G_2 to G_1 is 1/2, while phenomenology predicts $G_2/G_1\sim4$. In principle, the problem can be solved by introducing into (8.4) and (8.5) the separating functions $f_\sigma(t)$ $(\sigma=0,1,\ldots)$ corresponding to different localization of the meson states with spins $\sigma=0$ and $\sigma=1$. However, so as not to complicate our study of nonlocal corrections, we shall neglect this spin dependence.

The first three terms in (8.6) coincide with the expression for the Lagrangian (2.3) arising after linearization of the extended NJL model. After integrating by parts and discarding the surface terms, the last term in (8.6) can be rewritten as

$$\int d^4z\partial^\mu\bar{q}(z)\,\eta(z)\partial_\mu q(z)=-\int d^4z\bar{q}(z)[\partial^\mu\eta(z)\partial_\mu$$
$$+\eta(z)\partial^2]q(z). \tag{8.9}$$

Of course, we do not know the exact form of the function $f(t)$ for estimating the parameter α in Eq. (8.6). However, reasonable estimates can be obtained by using the fixed-separation approximation,[23] corresponding to the case where the constituent quarks in the meson are strongly localized on a scale h.

This strong localization can be taken into account by introducing the δ function $\delta((x-y)^2-h^2)$ into the integrand of the action $\mathcal{S}_{\rm int}$ (1.5). Then the action (1.5) takes the form

$$\mathcal{S}_{\rm int}=-i\,\frac{\kappa^2}{2}\int\int d^4x d^4y j^a_\mu(x) j^{a\mu}(y) D(x-y)\,\delta((x-y)^2-h^2), \qquad (8.10)$$

where the constant κ is introduced to give the correct dimension ($[\kappa]=m^{-1}$). After shifting the argument y by using the Lorentz-invariant operator

$$q(y)=\exp((y-x)_\mu\partial^\mu)q(x),$$

the effective action (8.10) can be rewritten as

$$\mathcal{S}_{\rm int}=-i\,\frac{\kappa^2 D(h)}{2}\int d^4x j^a_\mu(x) K(h,x) j^{a\mu}(x), \qquad (8.11)$$

where

$$K(h,x)=\int d^4y\,\exp((y-x)_\mu\partial^\mu)\,\delta((x-y)^2-h^2). \qquad (8.12)$$

Integrating over polar coordinates and expanding (8.12) in a series

$$K(h,x)=\pi^2h^2\sum_{n=0}^{\infty}\frac{1}{(2n)!}\,\frac{\Gamma\left(n+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma(n+2)}\,h^{2n}\Box^n$$

$$=\pi^2h^2\left[1+\frac{1}{8}\frac{\Box}{\tilde{\Lambda}^2}+O\left(\frac{\Box^2}{\tilde{\Lambda}^4}\right)\right], \qquad (8.13)$$

we can rewrite the action (8.11) as

$$\mathcal{S}_{\rm int}=-i\,\frac{9G}{16}\int d^4x\left[j^a_\mu(x)j^{a\mu}(x)+\frac{1}{8\tilde{\Lambda}^2}j^a_\mu(x)\times(\Box j^{a\mu}(x))\right]+O\left(\frac{\Box^2}{\tilde{\Lambda}^4}\right), \qquad (8.14)$$

where $G=\frac{8}{9}\pi^2\kappa^2h^2D(h)$.

After a Fierz transform, the action (8.14) is rewritten as

$$\mathcal{S}_{\rm int}=i\,\frac{9G}{16}\int d^4x\left(\bar{q}(x)\frac{\mathcal{M}^\theta}{2}q(x)\bar{q}(x)\frac{\mathcal{M}^\theta}{2}q(x)+\frac{1}{8\tilde{\Lambda}^2}\bar{q}(x)\frac{\mathcal{M}^\theta}{2}\Box[q(x)\bar{q}(x)]\frac{\mathcal{M}^\theta}{2}q(x)\right), \qquad (8.15)$$

where $\mathcal{M}^\theta$ is a tensor product of the form (1.7). The first term in (8.15) leads to the effective four-quark interaction of the NJL model, while the second term proportional to $1/\tilde{\Lambda}^2$ takes into account the effects of the finite size of the collective mesons. The second term in (8.15) can be transformed to

$$\mathcal{S}^2_{\rm int}=\frac{i}{16\tilde{\Lambda}^2}\int d^4x\bar{q}(x)[\partial^\mu\eta(x)\partial_\mu+\eta(x)\partial^2]q(x)+(\text{excitation terms}), \qquad (8.16)$$

where $\eta(x)$ is a combination of local collective fields (8.5), which are now given by (2.2) for $G_1=2G_2=G/4$. Comparing Eqs. (8.9) and (8.16), we obtain $\alpha=1/16$, which corresponds to the naive fixed-distance approximation in the bilocal model.

If the meson is treated as a bound $\bar{q}q$ system in the effective gluon potential, by analogy with the hydrogen atom in quantum mechanics, the corresponding nonrelativistic Schrödinger equation can be written as[71] ($\hbar=c=1$)

$$-\frac{1}{2m}\nabla^2\Psi(\mathbf{r})+[V(\mathbf{r})-E]\Psi(\mathbf{r})=0.$$

Here Ψ is the wave function of the internal motion; m is the reduced mass of the constituent quarks in the two-particle system, $m=(m_1m_2)/(m_1+m_2)$; $\mathbf{r}$ is the relative coordinate; $V(\mathbf{r})$ is the interaction potential; and E is the eigenvalue of the Hamiltonian. For a spherical potential the wave function Ψ is usually represented as the product of the radial function $R(r)$, where $r=|\mathbf{r}|$, and the spherical harmonics $Y_{lm}(\Theta,\phi)$: $\Psi(\mathbf{r})=R(r)Y_{lm}(\Theta,\phi)$. The Schrödinger equation for the radial function has the form

$$-u''(r)=2m\left[E-V(r)-\frac{l(l+1)}{2mr^2}\right]u(r),\quad u(r)=rR(r), \qquad (8.17)$$

with the boundary condition $u(0)=0$.

The results of lattice calculations of QCD[72] show that at large distances the effective quark–antiquark interaction can be approximated (neglecting Coulomb and other corrections) by a linearly growing potential:[11)]

$$V(r)=\sigma\cdot r, \qquad (8.18)$$

where $\sigma\approx0.27\ \mathrm{GeV}^2$ for $m_u=m_d=\mu=0.336$ GeV (Ref. 73). In this case the characteristic distance between a constituent quark and antiquark, $\langle r\rangle$, can easily be estimated using the virial theorem:

$$\langle r\rangle\equiv h=\frac{2E_1}{3\sigma}\approx0.68\ \mathrm{F}.$$

Here $E_1=2.238(\sigma^2/2\mu)^{1/3}$ is the ground-state energy. Henceforth we shall consider only the ground states ($l=0$), i.e., we shall discard excited states such as π^*, K^*, etc. After scaling Eq. (8.17) by introducing $\rho=r/h_0$ and $w(\rho)=u(r)$, we obtain $w''(\rho)+(\varepsilon-2\rho)w(\rho)=0$, where $\varepsilon=2m(2m\sigma)^{-2/3}E$. For large ρ the solutions behave like Airy functions, $w(\rho)\sim\mathrm{Ai}(\rho)$, decreasing exponentially to zero. The radial wave function $R(r)$, given by (8.17), falls off even more quickly. We can therefore expect small rms deviations Δr for the distance between constituent quarks.

An alternative estimate of the effective distance can be obtained by lattice modeling of QCD.[74] Here the correlation length, which is a measure of the range of the color forces, turns out to be considerably smaller: $h\sim0.22$ F. For numerical estimates we will use the value $h \sim 0.68$ F, obtained from the virial theorem. This will give us an upper limit on the estimates of nonlocal effects.

Using the values $\tilde{\Lambda}=0.28$ GeV and $\mu=0.336$ GeV, we find

$$\frac{\alpha\mu^2}{\tilde{\Lambda}^{22}} \approx 0.09. \tag{8.19}$$

We shall treat this quantity as a small expansion parameter in numerical estimates of nonlocal effects.

After integrating over quark fields, the total action arising from the generating functional (8.6) is written in a form analogous to (2.6):

$$\mathscr{A}(\Phi,\Phi^\dagger,V,A)=\int d^4z\left[-\frac{1}{4G_1}\mathrm{tr}(\Phi^\dagger\Phi)-\frac{1}{4G_2}\mathrm{tr}(V_\mu^2 + A_\mu^2)\right]-i\,\mathrm{Tr}'[\ln(i\hat{\tilde{\mathbf{D}}})].$$

Here the second term is the quark determinant of the Dirac operator $i\hat{\tilde{\mathbf{D}}}$, modified by nonlocal contributions. An expression for it can be obtained from (2.4) using the replacement

$$A_\mu^{R/L}\to A_\mu^{R/L}\left(1+\frac{\alpha}{\tilde{\Lambda}^2}\partial^2\right)+\frac{\alpha}{\tilde{\Lambda}^2}(\partial_\nu A_\mu^{R/L})\partial^\nu,$$

$$\Phi\to\Phi\left(1+\frac{\alpha}{\tilde{\Lambda}^2}\partial^2\right)+\frac{\alpha}{\tilde{\Lambda}^2}(\partial_\nu\Phi)\partial^\nu.$$

The heat-kernel method can also be used to calculate the modulus of the quark determinant for the Dirac operator $\hat{\tilde{\mathbf{D}}}$, modified by nonlocal contributions. In this case the operator $\hat{\tilde{\mathbf{D}}}^\dagger\hat{\tilde{\mathbf{D}}}$ takes the form

$$\hat{\tilde{D}}^\dagger\hat{\tilde{D}}=\beta\partial^2+\mu^2+2\Gamma_\mu\partial^\mu+\Gamma_\mu^2+a(x) + \frac{\alpha}{\tilde{\Lambda}^2}\left[b(x)\tilde{D}x;{}^{\wedge\dagger}\tilde{D}x;\gamma=\beta\partial^2+\mu^2+2\Gamma_\mu\partial^\mu + \Gamma_\mu^2+a(x)+\frac{\alpha}{\tilde{\Lambda}//2}[b(x)+Q_\alpha(x)\partial^\alpha+c(x)\partial^2 + 2(\Gamma_\mu\partial^2+\partial_\alpha\Gamma_\mu\partial^\alpha)\partial^\mu]+O\left(\frac{\alpha^2}{\tilde{\Lambda}^4}\right)\right.,$$

where $\beta=1+2\alpha\mu^2/\tilde{\Lambda}^2$. The combinations $a(x)$, $b(x)$, $c(x)$, and $Q_\alpha(x)$ do not contain differential operators acting on the quark fields:

$$a(x)=i\gamma^\mu(P_RD_\mu\Phi+P_L\bar{D}_\mu\Phi^\dagger)+P_R\mathscr{M}+P_L\bar{\mathscr{M}} + \frac{1}{4}[\gamma^\mu,\gamma^\nu]\Gamma_{\mu\nu},$$

$$b(x)=i\gamma^\mu[P_R(A_\mu^L\partial^2\Phi+\partial_\alpha A_\mu^L\partial^\alpha\Phi-\Phi\partial^2A_\mu^R - \partial_\alpha\Phi\partial^\alpha A_\mu^R)+P_L(A_\mu^R\partial^2\Phi^\dagger+\partial_\alpha A_\mu^R\partial^\alpha\Phi^\dagger - \Phi^\dagger\partial^2A_\mu^L-\partial_\alpha\Phi^\dagger\partial^\alpha A_\mu^L)]+P_R(\Phi^\dagger\partial^2\Phi + \partial_\alpha\Phi^\dagger\partial^\alpha\Phi)+P_L(\Phi\partial^2\Phi^\dagger+\partial_\alpha\Phi\partial^\alpha\Phi^\dagger) + \Gamma_\mu\partial^2\Gamma^\mu+(\partial_\alpha\Gamma_\mu)^2+\frac{1}{4}[\gamma^\mu,\gamma^\nu]([\partial_\alpha\Gamma_\mu,\partial^\alpha\Gamma_\nu] + \Gamma_\mu\partial^2\Gamma_\nu-\Gamma_\nu\partial^2\Gamma_\mu),$$

$$c(x)=a(x)+i\gamma^\mu[P_R(A_\mu^L\Phi-\Phi A_\mu^R)+P_L(A_\mu^R\Phi^\dagger - \Phi^\dagger A_\mu^L)]+P_R\mathscr{M}+P_L\bar{\mathscr{M}}+2\Gamma_\mu^2+\frac{1}{4}[\gamma^\mu,\gamma^\nu] \times[\Gamma_\mu,\Gamma_\nu],$$

$$Q_\alpha(x)=3\Gamma_\mu\partial_\alpha\Gamma^\mu+\partial_\alpha\Gamma_\mu\Gamma^\mu+\partial_\alpha a(x) + 2i\gamma^\mu[P_R(A_\mu^L\partial_\alpha\Phi-\Phi\partial_\alpha A_\mu^R)+P_L(A_\mu^R\partial_\alpha\Phi^\dagger - \Phi^\dagger\partial_\alpha A_\mu^L)]+2(P_R\Phi^\dagger\partial_\alpha\Phi+P_L\Phi\partial_\alpha\Phi^\dagger) + \frac{1}{2}[\gamma^\mu,\gamma^\nu](\Gamma_\mu\partial_\alpha\Gamma_\nu-\Gamma_\nu\partial_\alpha\Gamma_\mu).$$

The modified recursion equations for the heat-kernel coefficients $h_k(x)=h_k(x,y=x)$ have the form

$$2\frac{\alpha}{\tilde{\Lambda}^2}\Gamma_\mu t^\mu t^2h_{n+3}(x,y)+\frac{\alpha}{\tilde{\Lambda}^2}[t^2(2\Gamma_\mu+c(x)) + 2\Gamma_\mu(3t^\mu+2t^\mu t_\alpha\partial^\alpha+t^2\partial^\mu)+2\partial_\alpha\Gamma_\mu t^\mu t^\alpha]h_{n+2}(x,y) + \left[n+1+2t_\mu d^\mu+\frac{\alpha}{2\tilde{\Lambda}^2}\quad(+4c(x)(1+t_\alpha\partial^\alpha) - 8\mu^2t_\alpha d^\alpha+2Q_\alpha(x)t^\alpha+2\partial_\alpha\Gamma_\mu(t^\mu\partial^\alpha+t^\alpha\partial^\mu) + 4\Gamma_\mu(2\partial^\mu+2t_\alpha\partial^\alpha\partial^\mu+t^\mu\partial^2))\right]h_{n+1}(x,y)+\left[a(x) + d_\mu d^\mu+\frac{\alpha}{\tilde{\Lambda}^2}(b(x)+Q_\alpha(x)\partial^\alpha+2(\Gamma_\mu\partial^2+\partial_\alpha\Gamma_\mu\partial^\alpha)\partial^\mu + c(x)\partial^2)\right]h_n(x,y)=0,$$

where the differentiation operator ∂_μ acts only on x. In the case $V_\mu=A_\mu=0$ the recursion relations for the heat-kernel coefficients reduce to the equation

$$\frac{\alpha}{\tilde{\Lambda}^2}t^2\tilde{c}(x)h_{n+2}(x,y)+\left\{n+1+2t_\mu\partial^\mu+\frac{\alpha}{\tilde{\Lambda}^2}[2(\tilde{a}(x) + \tilde{c}(x))(1+t_\mu\partial^\mu)-4\mu^2t_\mu\partial^\mu+t_\mu(\partial^\mu\tilde{a}(x) + 2\tilde{Q}^\mu(x))]\right\}h_{n+1}(x,y)+\left\{\tilde{a}(x)+\partial^2+\frac{\alpha}{\tilde{\Lambda}^2}[\tilde{b}(x) + (\partial_\mu\tilde{a}(x)+2\tilde{Q}_\mu(x))\partial^\mu+(\tilde{a}(x) + \tilde{c}(x))\partial^2)\right\}h_n(x,y)=0, \tag{8.20}$$

where

$$\tilde{a}(x)=i\gamma^{\mu}(P_R\partial_\mu\Phi+P_L\partial_\mu\Phi^\dagger)+P_R\mathcal{M}+P_L\bar{\mathcal{M}},$$

$$\tilde{b}(x)=P_R(\Phi^\dagger\partial^2\Phi+\partial_\mu\Phi^\dagger\partial^\mu\Phi)+P_L(\Phi\partial^2\Phi^\dagger+\partial_\mu\Phi\partial^\mu\Phi^\dagger),$$

$$\tilde{c}(x)=P_R\mathcal{M}+P_L\bar{\mathcal{M}},$$

$$\tilde{Q}_\mu(x)=P_R\Phi^\dagger\partial_\mu\Phi+P_L\Phi\partial_\mu\Phi^\dagger.$$

From (8.20) we can obtain expressions for the heat-kernel coefficients $h_0,\dots,h_3$, modified by nonlocal contributions:

$$h_0(x)=1,$$

$$\mathrm{tr}'[h_1(x)]=-\mathrm{tr}'\left[\tilde{a}+\frac{\alpha}{\tilde{\Lambda}^2}(\tilde{b}-\tilde{a}(\tilde{a}+\tilde{c}))\right]+O\left(\frac{\alpha^2}{\tilde{\Lambda}^4}\right),$$

$$\mathrm{tr}'[h_2(x)]=\mathrm{tr}'\left[\frac{1}{2}\tilde{a}^2+\frac{\alpha}{\tilde{\Lambda}^2}\left(\tilde{a}\tilde{b}-\frac{2}{3}\tilde{a}^2(\tilde{a}+\tilde{c})+\frac{5}{12}(\partial_\mu\tilde{a})^2-\frac{1}{12}\partial_\mu\tilde{a}\partial^\mu\tilde{c}-\tilde{a}\partial^\mu\tilde{Q}_\mu\right)\right]+O\left(\frac{\alpha^2}{\tilde{\Lambda}^4}\right),$$

$$\mathrm{tr}'[h_3(x)]=-\mathrm{tr}'\left\{\frac{1}{6}\tilde{a}^3-\frac{1}{12}(\partial_\mu\tilde{a})^2+\frac{\alpha}{\tilde{\Lambda}^2}\left[\frac{1}{2}\tilde{a}^2\tilde{b}-\frac{1}{4}\tilde{a}^3(\tilde{a}+\tilde{c})-\tilde{a}^2\left(\frac{3}{10}\partial^2\tilde{a}+\frac{1}{2}\partial^\mu\tilde{Q}_\mu+\frac{2}{3}\partial^2\tilde{c}\right)-\tilde{a}\left(\frac{5}{6}(\partial_\mu\tilde{a})^2+\frac{1}{15}\partial^2\tilde{a}\tilde{c}+\frac{2}{5}\partial_\mu\tilde{a}\partial^\mu\tilde{c}+\frac{11}{30}\partial_\mu\tilde{c}\partial^\mu\tilde{a}+\frac{1}{20}(\partial^2\partial^2\tilde{c}+\tilde{c}\partial^2\tilde{a})+\frac{1}{6}(\partial^2\partial^\mu\tilde{Q}_\mu-\tilde{Q}_\mu\partial^\mu\tilde{a}+\partial^\mu\tilde{a}\tilde{Q}_\mu-\partial^2\tilde{b})\right)-\frac{1}{15}(\partial^2\tilde{c}\partial^2\tilde{a}-\tilde{c}(\partial_\mu\tilde{a})^2-(\partial^2\tilde{a})^2)-\frac{1}{18}\mu^2(\partial_\mu\tilde{a})^2\right]\right\}+O\left(\frac{\alpha^2}{\tilde{\Lambda}^4}\right).$$

The expressions for h_1, h_2, and h_3 are sufficient for calculating the nonlocal corrections to the effective bosonized Lagrangian including p^4 terms (nonlocal corrections to the coefficient h_4 contribute beginning with p^6 terms).

Calculations of the quark determinant taking into account nonlocal effects[23] lead to a modified expression for the constant F_0:

$$F_0^2=\frac{N_c\mu^2}{4\pi^2}\left[y-\frac{4\pi^2\langle\bar{q}q\rangle}{\mu^3N_c}\frac{\alpha\mu^2}{\tilde{\Lambda}^2}\right], \tag{8.21}$$

where the first term corresponds to the local limit, while the second is the nonlocal correction. For the meson mass matrix $\chi=\mathrm{diag}(\chi_u^2,\chi_d^2,\dots,\chi_n^2)$ we obtain

$$\chi_i^2=\frac{N_c\mu m_i^0}{2\pi^2F_0^2}(\Lambda^2e^{-\mu^2/\Lambda^2}-\mu^2y)=-\frac{2m_i^0\langle\bar{q}q\rangle}{F_0^2}. \tag{8.22}$$

Moreover, the structure coefficients L_i of the minimal part of the Gasser–Leutwyler effective p^4 Lagrangian are given by the relations $L_1-L_2/2=L_4=0$ and

$$L_2=\frac{N_c}{16\pi^2}\frac{1}{12}\left(1+2\frac{\alpha\mu^2}{\tilde{\Lambda}^2}\right),$$

$$L_3=-\frac{N_e}{16\pi^2}\frac{1}{6}\left(1+5(1-y)\frac{\alpha\mu^2}{\tilde{\Lambda}^2}\right),$$

$$L_5=\frac{N_c}{16\pi^2}x\left[y-1-\frac{28}{3}\frac{\alpha\mu^2}{\tilde{\Lambda}^2}\right]. \tag{8.23}$$

The nonlocal corrections to the nonminimal part of the effective meson p^4 Lagrangian can be estimated in a similar way. Here we shall confine ourselves to only the modified expressions for the structure coefficients L_9 and L_{10}:

$$L_9=\frac{N_c}{16\pi^2}\frac{1}{3}\left(1+\frac{21y-26}{6}\frac{\alpha\mu^2}{\tilde{\Lambda}^2}\right),$$

$$L_{10}=-\frac{N_c}{16\pi^2}\frac{1}{6}\left(1+\frac{15y-10}{3}\frac{\alpha\mu^2}{\tilde{\Lambda}^2}\right). \tag{8.24}$$

It should be noted in particular that nonlocal corrections do not affect the meson mass matrix. The nonlocal corrections to the structure coefficients L_2, L_3, L_9, and L_{10} [see Eqs. (8.23) and (8.24)] do not exceed 15–20% compared to the local limit. The coefficient L_5 is the most sensitive to nonlocal corrections, as a result of which nonlocal effects must strongly affect the description of the splitting of the decay constants F_π and F_K. It should be emphasized that the estimates obtained using the value $h=0.68$ F from the Schrödinger equation for a linear growing potential give upper limits for the nonlocal corrections which are too high. If we use the value $h\sim0.22$ F from lattice QCD,[74] the nonlocal corrections will be less than 5%. This result explains why, in spite of its incompleteness, the bosonization of the NJL model leads to a correct description of low-energy meson processes in terms of effective chiral Lagrangians. The smallness of the calculated nonlocal corrections shows that the NJL model, as a local model of effective four-quark interactions, is actually a reasonable physical approximation describing the chiral quark dynamics.

CONCLUSION

QCD-motivated bilocal models of effective quark interactions and the various modifications of the extended NJL model arising from them in the local limit are fruitful approaches to the study of the internal quark structure of hadrons. In this review we have confined ourselves to the detailed discussion of only those aspects of the bosonization of the NJL model which are related to obtaining chiral meson Lagrangians in higher orders of the momentum expansion. Many other questions lying outside our treatment, including the Bethe–Salpeter equations for bound $\bar{q}q$ states, diquarks, and the soliton model of baryons, are discussed in detail in Refs. 12, 27, 30, and 31 and references therein.

A cutoff at large momenta is introduced into the ordinary NJL model to regularize the ultraviolet divergences in quark loops with external meson fields arising in the calculation of the quark determinant. In addition, in this approach the momentum dependence at quark–meson vertices (nonlocality), which reflects the composite structure of hadrons in more detail, is neglected, as is quark confinement. Nevertheless, the bosonization of the NJL model leads to the correct phenomenology of low-energy meson processes not only in the tree p^2 approximation, which reproduces the current-algebra results, but also in the higher orders of perturbation theory taking into account unitarity corrections due to meson loops. It is this which is the leading argument in favor of the NJL model as a local, low-energy approximation of QCD.

In discussing the phenomenological status of bosonized Lagrangians, we have restricted ourselves to the pseudoscalar-meson sector. We showed that vector, axial-vector, and scalar mesons can be excluded from explicit consideration by integrating over the corresponding resonance degrees of freedom in the generating functional of the bosonized NJL model. As a rule, modification of the structure coefficients of effective Lagrangians after resonance reduction leads to a significant improvement of the description of the experimental data on low-energy meson processes in both order p^4 and order p^6 of chiral perturbation theory with bosonized Lagrangians. Nevertheless, it should be noted that the description of processes involving on-shell vector and axial-vector mesons apparently lies at the limit of the applicability of the momentum expansion in the chiral theory.

The difficulties due to the limited applicability of the momentum expansion already arise in describing the decays of η and η' mesons. There it is also necessary to include $U(1)$-symmetry breaking, for example, by introducing an additional six-quark interaction induced by instantons and associated with the 't Hooft determinant.[80] This extension of the NJL model has been used in Ref. 81 to obtain from the solution of the Bethe–Salpeter equations for bound $\bar{q}q$ states the mass spectra of pseudoscalar, scalar, vector, and axial-vector mesons. However, in the approach of effective chiral Lagrangians, for formalizing the solution of the $U(1)$ problem it is more convenient to use the method of Ref. 82 based on the inclusion of the gluon anomaly[83] by introducing gluon fields in the form of a topological charge density into the chiral Lagrangian.

We should specially emphasize the well known fact that the standard NJL model is not in a position to correctly describe the dynamics of on-shell low-lying meson resonances.[84] In this model, for example, vector mesons can exist on the mass shell only if the constituent quark mass exceeds 385 MeV, but even then there is a large unphysical $\bar{q}q$ continuum in the spectral functions. For smaller values of the constituent quark mass, vector mesons cannot exist even as resonances in the $\bar{q}q$ continuum.[85] The authors of Ref. 86 arrived at the same conclusion. They studied the properties of vector and axial-vector mesons at finite temperature and density in the NJL model. They showed that with the standard choice of parameters, the model predicts a large width for the decay $\rho\rightarrow q\bar{q}$, of the same order of magnitude as the ρ-meson mass. As a result, the ρ meson is extremely unstable. These facts associated with the absence of confinement in the NJL model represent a serious difficulty with this model, because they practically exclude the possibility of interpreting collective meson excitations in the vector and axial-vector sector as real vector and axial-vector mesons. This problem can only be solved by going beyond the standard NJL model to include quark confinement.

The bilocal approach can be used to obtain various extensions of the NJL model which include both nonlocality and modeling of confinement. For example, the authors of Ref. 21 have studied the modeling of quark confinement within the modified NJL model arising from the bilocal action (1.6), using a special ansatz for the nonperturbative gluon propagator. In momentum space this ansatz contains the sum of two terms: a constant corresponding to the usual local interaction of the NJL model, and a δ function effectively including the $1/q^4$ behavior of the gluon propagator in the confinement region. The introduction of the δ function modifies the Schwinger–Dyson equation and leads to a behavior of the running constituent quark mass $m(q^2)$ which ensures the absence of poles in its propagator.

The extended treatment of the NJL model developed in Ref. 22 gives a deeper understanding of the role of confinement in the bosonization of QCD. The authors of these studies, in considering quark loops with external meson fields, assumed locality of the quark–meson vertices, while describing the quark propagators by internal analytic functions ensuring not only quark confinement, but also ultraviolet convergence of quark loops. In Ref. 87 a formulation of the NJL model with separable interaction was proposed which includes nonlocality by means of suitable form factors at four-quark vertices.

In the last section of this review we used a semiphenomenological approach to study nonlocal effects associated with the contribution of confinement to the bilocal action. It was shown that nonlocal corrections to the structure coefficients of the bosonized p^4 Lagrangian of the NJL model are quite small.[12)] This result shows that the interaction dynamics of pseudoscalar mesons at low energies is insensitive to quark confinement and is determined by chiral symmetry and its breaking. It can therefore be expected that the inclusion of the modeling of confinement in a specific, extended version of the NJL model in order to eliminate the problems associated with the description of the dynamics of on-shell vector and axial-vector mesons will not seriously affect the description of the pseudoscalar sector.

The functional methods used in the bosonization of strong interactions in the NJL model can also be used to bosonize effective four-quark nonleptonic weak interactions with change of strangeness $|\Delta S|=1$. For this one uses the Green-function generating functional introduced in Ref. 88

for the quark currents and their densities, products of which enter into the weak four-quark Lagrangian. In this approach, $(V-A)$ and $(S-P)$ meson currents arise as a result of variation of the quark determinant of the NJL model with respect to additional external sources associated with the corresponding quark currents and their densities. The additional external sources arise in the quark determinant as a result of redefinition of the Dirac operator. As a result, the terms of the quark determinant contributing to the bosonized strong Lagrangian of order p^{2n} in the momentum expansion generate $(V-A)$ and $(S-P)$ meson currents of order p^{2n-1} and p^{2n-2}, respectively.

As shown, in particular, in Ref. 88, estimates of direct CP violation in the decays $K\to 2\pi$ and $K\to 3\pi$, arising as a difference effect from the interference of different isotopic amplitudes, turn out to be very sensitive to the higher orders of chiral perturbation theory. This is also true of the phases of the pion final-state interactions in the decays $K\to 2\pi$ and $K\to 3\pi$ and the quadratic parameters of the slope of the Dalitz plots for $K\to 3\pi$ decays. More accurate measurements of the slope parameters of $K^{\pm}\to\pi^{\pm}\pi^{0}\pi^{0}$ decays are being made in two experiments currently running at the U-70 accelerator in Protvino.[89] Accurate measurements of the slope parameters are needed to check the new analysis of nonleptonic K decays in the chiral theory in order to fix the model dependences in calculations of the meson matrix elements, to phenomenologically determine the Wilson coefficients, and to decrease the model uncertainties in predictions of the observed effects of direct CP violation. At present we are performing calculations of the amplitudes of $K\to 2\pi$ and $K\to 3\pi$ decays in order p^6 of the chiral theory which are more complete than those in Ref. 88. These calculations include the $(V-A)$ and $(S-P)$ meson currents generated by the higher-order bosonized Lagrangians discussed in this review.

APPENDIX: THE p^6 CONTRIBUTIONS OF THE HEAT-KERNEL COEFFICIENTS

Here we give the initial expressions [before using equivalence transformations following from the properties of the covariant derivatives (5.4) and the equations of motion] for the p^6 contributions of the finite part of the bosonized Lagrangian, which arises from the heat-kernel coefficients h_3, h_4, h_5, and h_6 after calculating the trace over Dirac matrices:

$$\mathscr{L}_{\rm fin}^{(p^6)}=-\frac{N_c}{32\pi^2\mu^6}\,{\rm tr}\Big(\mu^4H_3^{p^6}+\mu^2H_4^{p^6}+2H_5^{p^6}+\frac{6}{\mu^2}H_6^{p^6}\Big). \tag{A1}$$

The corresponding contributions are

$$\begin{aligned}
H_3^{p^6}=&-\frac{1}{3}(\mathscr{M}^3+\bar{\mathscr{M}}^3)+\frac{1}{6}[(D'_\mu\mathscr{M})^2+(\bar{D}'_\mu\bar{\mathscr{M}})^2]\\
&+\frac{1}{3}[\mathscr{M}(F^L_{\mu\nu})^2+\bar{\mathscr{M}}(F^R_{\mu\nu})^2]-\frac{1}{3}(F^L_{\mu\nu}F^{L^{\mu\alpha}}F^{L^\nu}{}_\alpha\\
&+F^R_{\mu\nu}F^{R^{\mu\alpha}}F^{R^\nu}{}_\alpha)-\frac{1}{54}[(F^{L^\mu}{}_{\mu\nu})^2+(F^{R^\mu}{}_{\mu\nu})^2]\\
&-\frac{41}{540}[(F^L_{\mu\nu\alpha})^2+(F^R_{\mu\nu\alpha})^2],
\end{aligned}$$

$$\begin{aligned}
H_4^{p^6}=&-\frac{1}{6}\Big[D'^2\mathscr{M}D_\mu\Phi\bar{D}^\mu\Phi^\dagger\\
&+\bar{D}'_\mu\bar{\mathscr{M}}(\bar{D}^\nu\bar{D}^\mu\Phi^\dagger D_\nu\Phi+\bar{D}_\nu\Phi^\dagger D^\mu D^\nu\Phi)\\
&+\bar{D}'^2\bar{\mathscr{M}}\bar{D}_\mu\Phi^\dagger D^\mu\Phi+D'_\mu\mathscr{M}(D^\nu D^\mu\Phi\bar{D}_\nu\Phi^\dagger\\
&+D_\nu\Phi\bar{D}^\mu\bar{D}^\nu\Phi^\dagger)-\frac{1}{6}[\mathscr{M}(D_\mu D_\nu\Phi\bar{D}^\mu\bar{D}^\nu\Phi^\dagger\\
&+D_\mu\Phi\bar{D}^2\bar{D}^\mu\Phi^\dagger+D^2D_\mu\Phi\bar{D}^\mu\Phi^\dagger)\\
&+\bar{\mathscr{M}}(\bar{D}_\mu\bar{D}_\nu\Phi^\dagger D^\mu D^\nu\Phi\\
&+\bar{D}_\mu\Phi^\dagger D^2D^\mu\Phi+\bar{D}^2\bar{D}_\mu\Phi^\dagger D^\mu\Phi)]\\
&-\frac{1}{3}(\mathscr{M}D_\mu\Phi\bar{\mathscr{M}}\bar{D}^\mu\Phi^\dagger\\
&+\mathscr{M}^2D_\mu\Phi\bar{D}^\mu\Phi^\dagger+\bar{\mathscr{M}}^2\bar{D}_\mu\Phi^\dagger D^\mu\Phi)\\
&-\frac{7}{180}D^2D_\mu\Phi\bar{D}^2\bar{D}^\mu\Phi^\dagger\\
&+\frac{1}{180}D_\mu D_\nu D_\alpha\Phi\bar{D}^\mu\bar{D}^\nu\bar{D}^\alpha\Phi^\dagger\\
&+\frac{1}{3}\Big(\frac{2}{5}D_\mu\Phi F^R_{\nu\alpha}\bar{D}^\mu\Phi^\dagger F^{L^{\nu\alpha}}-D_\mu\Phi F^R_{\nu\alpha}\bar{D}^\nu\Phi^\dagger F^{L^{\mu\alpha}}\\
&+D^\mu\Phi F^R_{\mu\nu}\bar{D}_\alpha\Phi^\dagger F^{L^{\nu\alpha}}\Big)+\frac{1}{10}[D_\mu\Phi\bar{D}^\mu\Phi^\dagger(F^L_{\nu\alpha})^2\\
&+\bar{D}_\mu\Phi^\dagger D^\mu\Phi(F^R_{\nu\alpha})^2]-\frac{1}{3}[D_\mu\Phi\bar{D}^\nu\Phi^\dagger[F^{L^{\mu\alpha}},F^L_{\nu\alpha}]\\
&+\bar{D}_\mu\Phi^\dagger D^\nu\Phi[F^{R^{\mu\alpha}},F^R_{\nu\alpha}]]+\frac{1}{6}(D^\mu\Phi\bar{D}^\nu\Phi^\dagger F^{L^\alpha}{}_{\alpha\mu\nu}\\
&+\bar{D}_\mu\Phi^\dagger D_\nu\Phi F^{R^\alpha}{}_{\alpha\mu\nu})+\frac{1}{6}[(D^\alpha D^\mu\Phi\bar{D}_\alpha\bar{D}^\nu\Phi^\dagger\\
&+D^2D^\mu\Phi\bar{D}^\nu\Phi^\dagger\\
&+D^\mu\Phi\bar{D}^2\bar{D}^\nu\Phi^\dagger)F^L_{\mu\nu}+(\bar{D}^\alpha\bar{D}^\mu\Phi^\dagger D_\alpha D^\nu\Phi\\
&+\bar{D}^2\bar{D}^\mu\Phi^\dagger D^\nu\Phi+\bar{D}^\mu\Phi^\dagger D^2D^\nu\Phi)F^R_{\mu\nu}]\\
&+\frac{1}{180}[(D^\mu D^\nu D^\alpha\Phi\bar{D}_\alpha\Phi^\dagger\\
&-D_\alpha\Phi\bar{D}^\mu\bar{D}^\nu\bar{D}^\alpha\Phi^\dagger)F^L_{\mu\nu}+(\bar{D}^\mu\bar{D}^\nu\bar{D}^\alpha\Phi^\dagger D_\alpha\Phi
\end{aligned}$$

$$-\bar{D}_\alpha\Phi^\dagger D^\mu D^\nu D^\alpha\Phi)F^R_{\mu\nu}]+\frac{1}{6}[(D^\nu D^\alpha\Phi\bar{D}^\mu\Phi^\dagger$$

$$-D^\mu\Phi\bar{D}^\nu\bar{D}^\alpha\Phi^\dagger)F^L_{\nu\alpha\mu}+(\bar{D}^\nu\bar{D}^\alpha\Phi^\dagger D^\mu\Phi$$

$$-\bar{D}^\mu\Phi^\dagger D^\nu D^\alpha\Phi)F^R_{\nu\alpha\mu}]-\frac{1}{72}[(D^\alpha D^\mu\Phi\bar{D}_\mu\Phi^\dagger$$

$$-D_\mu\Phi\bar{D}^\alpha\bar{D}^\mu\Phi^\dagger)F^{L^\nu}{}_{\nu\alpha}$$

$$+(\bar{D}^\alpha\bar{D}^\mu\Phi^\dagger D_\mu\Phi-\bar{D}_\mu\Phi^\dagger D^\alpha D^\mu\Phi)F^{R^\nu}{}_{\nu\alpha}]$$

$$+\frac{1}{3}[(\mathscr{M}D^\mu\Phi\bar{D}^\nu\Phi^\dagger+D^\mu\Phi\bar{D}^\nu\Phi^\dagger\mathscr{M}$$

$$+D^\mu\Phi\bar{\mathscr{M}}\bar{D}^\nu\Phi^\dagger)F^L_{\mu\nu}$$

$$+(\bar{\mathscr{M}}\bar{D}^\mu\Phi^\dagger D^\nu\Phi+\bar{D}^\mu\Phi^\dagger D^\nu\Phi\bar{\mathscr{M}}$$

$$+\bar{D}^\mu\Phi^\dagger\mathscr{M}D^\nu\Phi)F^R_{\mu\nu},$$

$$H_5^{p^6}=\frac{1}{20}[(D_\mu\Phi\bar{D}_\nu\Phi^\dagger-D_\nu\Phi\bar{D}_\mu\Phi^\dagger)D^\alpha D^\mu\Phi\bar{D}_\alpha\bar{D}^\nu\Phi^\dagger$$

$$+(\bar{D}_\mu\Phi^\dagger D_\nu\Phi-\bar{D}_\nu\Phi^\dagger D_\mu\Phi)\bar{D}^\alpha\bar{D}^\mu\Phi^\dagger D_\alpha D^\nu\Phi$$

$$-D_\mu\Phi\bar{D}^\mu\Phi^\dagger D_\nu D_\alpha\Phi\bar{D}^\nu\bar{D}^\alpha\Phi^\dagger$$

$$-\bar{D}_\mu\Phi^\dagger D^\mu\Phi\bar{D}_\nu\bar{D}_\alpha\Phi^\dagger D^\nu D^\alpha\Phi]$$

$$+\frac{1}{30}[(D_\mu\Phi\bar{D}_\nu\bar{D}_\alpha\Phi^\dagger$$

$$-D_\alpha\Phi\bar{D}_\nu\bar{D}_\mu\Phi^\dagger)D^\mu\Phi\bar{D}^\nu\bar{D}^\alpha\Phi^\dagger$$

$$+(\bar{D}_\mu\Phi^\dagger D_\nu D_\alpha\Phi$$

$$-\bar{D}_\alpha\Phi^\dagger D_\nu D_\mu\Phi)\bar{D}^\mu\Phi^\dagger D^\nu D^\alpha\Phi$$

$$-D_\mu\Phi\bar{D}^\nu\bar{D}^\mu\Phi^\dagger D^\alpha\Phi\bar{D}_\nu\bar{D}_\alpha\Phi^\dagger$$

$$-\bar{D}_\mu\Phi^\dagger D^\nu D^\mu\Phi\bar{D}^\alpha\Phi^\dagger D_\nu D_\alpha\Phi)]$$

$$-\frac{1}{12}[\mathscr{M}((D_\mu\Phi\bar{D}^\mu\Phi^\dagger)^2$$

$$-(D_\mu\Phi\bar{D}_\nu\Phi^\dagger)^2+D_\mu\Phi\bar{D}_\nu\Phi^\dagger D^\nu\Phi\bar{D}^\mu\Phi^\dagger)$$

$$+\bar{\mathscr{M}}((\bar{D}_\mu\Phi^\dagger D^\mu\Phi)^2-(\bar{D}_\mu\Phi^\dagger D_\nu\Phi)^2$$

$$+\bar{D}_\mu\Phi^\dagger D_\nu\Phi\bar{D}^\nu\Phi^\dagger D^\mu\Phi)]+\frac{1}{12}[(\{D_\alpha\Phi\bar{D}^\alpha\Phi^\dagger,$$

$$D^\mu\Phi\bar{D}^\nu\Phi^\dagger\}+D_\alpha\Phi\bar{D}^\mu\Phi^\dagger$$

$$+D^\mu\Phi\bar{D}_\alpha\Phi^\dagger(D^\alpha\Phi\bar{D}^\nu\Phi^\dagger-D^\nu\Phi\bar{D}^\alpha\Phi^\dagger)$$

$$+D_\alpha\Phi\bar{D}^\mu\Phi^\dagger(D^\nu\Phi\bar{D}^\alpha\Phi^\dagger-D^\alpha\Phi\bar{D}^\nu\Phi^\dagger))F^L_{\mu\nu}$$

$$+(\{\bar{D}_\alpha\Phi^\dagger D^\alpha\Phi,\bar{D}^\mu\Phi^\dagger D^\nu\Phi\}$$

$$+\bar{D}^\mu\Phi^\dagger D_\alpha\Phi(\bar{D}^\alpha\Phi^\dagger D^\nu\Phi$$

$$-\bar{D}^\nu\Phi^\dagger D^\alpha\Phi)$$

$$+\bar{D}_\alpha\Phi^\dagger D^\mu\Phi(\bar{D}^\nu\Phi^\dagger D^\alpha\Phi$$

$$-\bar{D}^\alpha\Phi^\dagger D^\nu\Phi))F^R_{\mu\nu}],$$

$$H_p^{p^6}=-\frac{1}{90}(D_\mu\Phi\bar{D}^\mu\Phi^\dagger)^3$$

$$+\frac{1}{30}D_\mu\Phi\bar{D}^\mu\Phi^\dagger(D_\alpha\Phi\bar{D}_\nu\Phi^\dagger)^2$$

$$-\frac{1}{60}D_\mu\Phi\bar{D}^\mu\Phi^\dagger D_\nu\Phi\bar{D}_\alpha\Phi^\dagger D^\alpha\Phi\bar{D}^\nu\Phi^\dagger$$

$$+\frac{1}{180}D_\mu\Phi\bar{D}_\nu\Phi^\dagger D_\alpha\Phi\bar{D}^\mu\Phi^\dagger D^\nu\Phi\bar{D}_\alpha\Phi^\dagger$$

$$-\frac{1}{60}D_\mu\Phi\bar{D}_\nu\Phi^\dagger D^\mu\Phi\bar{D}_\alpha\Phi^\dagger D^\nu\Phi\bar{D}^\alpha\Phi^\dagger.$$

1)Here and everywhere below in this section where we do not use explicit super- and subscripts, there is understood to be a covariant summation over repeated Greek indices with the standard metric $(1,-1,-1,-1)$.

2)The difference from our notation for the tensors $F^{L,R}$ should be noted: $F^{L,R}=-iF^{R,L}_{\text{G\&L}}$.

3)After field transformations, the Lagrangians $\mathscr{L}_n(U)$ also contain derivatives of the fields V of order higher than p^{2n}.

4)We recall that the definition of the Dirac operator (2.4) involves the combination $\Phi+m_0$, where m_0 is a quantity of order p^2.

5)In the more general treatment of Ref. 66, the nonanomalous part of the effective p^6 Lagrangian includes a much larger number of linearly independent terms. However, in the particular case of bosonization of the NJL model that we are considering, only 65 of the structure coefficients turn out to be nonzero.

6)The Gaussian part of the effective action (2.6), which is quadratic in the fields Φ, V_μ, and A_μ, and also the chiral anomalies are not invariant under chiral rotations (2.8).

7)The pseudoscalar mass term arises in the effective Lagrangian (5.2) from the divergent part of the quark determinant.

8)Using the gap equation, it can be shown that the two expressions for χ in (6.10) are equivalent for $\mu^2/\Lambda^2\ll 1$.

9)For simplicity, we have discarded terms with the coefficient L_4, which is equal to zero in the NJL model.

10)In our approach, such contributions do not arise even when higher-order corrections to the static equations of motion are included, owing to the cancellation of the corresponding anomalous contributions after chiral rotation in the unitary gauge. In our opinion, the exact coincidence of the contributions of anomalous and nonanomalous vector exchanges noted in Ref. 79 is an indication that there may be double counting.

11)Here we neglect screening by virtual $\bar{q}q$ pairs at very large distances.

12)In some cases the contributions to the amplitudes of meson processes from nonlocal corrections to the structure coefficients of the p^4 Lagrangian can turn out to be comparable to the contributions of the p^6 Lagrangian.

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